Polya's Random Walk Theorem

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Abstract

Polya's random walk theorem states that a simple random walk on a d -dimensional lattice is recurrent for dimensions $d = 1, 2$ and transient for dimensions $d \geq 3$. In this paper we discuss the direct counting proof for this theorem, and examine the recurrence properties of random walks in Z_1 and Z_2, and contrast them with the transience property observed in Z_3.

Keywords: probability, random walks, lattice path, recurrent, transient

1 Introduction

Consider the integers in the one dimensional plane. Suppose there is a walker standing at the origin that starts flipping a coin. Every time the coin lands on tails, the walker takes a step to the left, and for every land on heads, the walker takes a step to the right. Assuming that the walker keeps repeating this process infinitely, what is the probability of the walker returning to zero?

Polya's random walk theorem, first proven by George Polya in 1921, answers the question and characterizes the behavior of random walks on infinite lattice paths of all dimensions:

Polya's Random Walk. *The simple random walk on $\mathbb{Z}^d$ is recurrent in dimensions $d = 1, 2$ and transient in dimensions $d \geq 3$.*

Since Polya's 1921 paper, this theorem became a key example in probability theory, leading to lots of research on random walks that have later been used in a variety of fields, including computer science, physics, ecology, and economics.

This paper involves two main sections: proving transience for dimension $d = 3$, and proving recurrence for dimensions $d = 1, 2$.

2 Definitions

We start by outlining the necessary background knowledge and concepts required to effectively comprehend and follow the proof.

2.1 Probability Review

Probability P is a fundamental concept in the field of mathematics that quantifies the likelihood of a particular event E occurring in the sample space

Ω. The ***sample space*** Ω is the set of all possible outcomes of an experiment, and an ***event*** E is a subset of Ω.

We can find the probability of an event E by dividing the number of ways that event can happen by the total number of outcomes. The result is expressed as a numerical value ranging from 0 to 1, where 0 signifies impossibility and 1 represents certainty.

In the context of probability, ***independent events*** are those whose occurrence does not influence the likelihood of other events happening. To formally define independent events, consider two events A and B. Events A and B are said to be independent if and only if the probability of both events occurring simultaneously is equal to the product of their individual probabilities, i.e., $P(A \cap B) = P(A)P(B)$.

A ***random variable*** is a mathematical function that assigns a numerical value to each outcome of a random experiment. It serves as a bridge between the abstract concept of probability and real-world applications by transforming uncertain events into measurable quantities.

The ***expected value***, also known as the mean or average, is a measure of the central tendency of a random variable. It represents the long-term average outcome of an experiment conducted many times under the same conditions. The expected value can be calculated by taking the sum of all possible values of a random variable, each multiplied by its corresponding probability, i.e., $\mathbb{E} = \sum P(X) \cdot X$ such that X is a random variable.

Variance, on the other hand, is a measure of dispersion or spread of a random variable. It quantifies the degree to which the individual values of a random variable deviate from its expected value. In essence, variance helps us understand the level of uncertainty or variability associated with a random variable. A low variance indicates that the outcomes are closely clustered around the expected value, whereas a high variance suggests greater variability and unpredictability.

In summary, probability is the cornerstone of understanding uncertain events and making informed predictions. Independent events allow us to assess the likelihood of multiple events occurring simultaneously without influencing each other. Random variables connect probability theory to real-world applications by quantifying uncertainty, while expected value and variance help us analyze the central tendency and dispersion of random variables. Together, all these concepts are fundamental for the counting proof of Polya's Random Walk Theorem.

2.2 Random Walks Review

The symbol $\mathbb{Z}$ denotes the set of all integers, i.e. the set $\mathbb{Z}^d$, often called the ***d-dimensional lattice***, is the set of all d-tuples of all integers. When $d = 1$, our lattice is just an infinite line divided into segments of unit length. When $d = 2$, our lattice looks like an infinite network of streets, which justifies why we describe the random motion of the wandering point as a "walk". When $d = 3$, the lattice looks like an infinite "jungle gym", as you can observe in the following image.

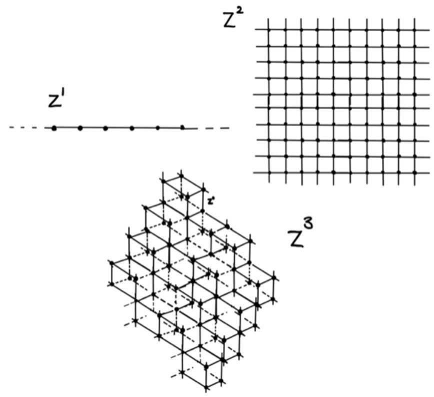

Figure 1

A ***simple random walk*** was one of the first chance processes studied in probability, which continues to play an important role in probability theory and its applications. It is defined as a walk starting at the origin, in which for each step the walker moves to a randomly selected neighboring point on $\mathbb{Z}^d$, with equal probability of making a step in any direction, and all steps having equal size.

Assuming S_n is the walker's location at step n, a simple random walk is ***recurrent*** if one of the following holds:

a) $P(\ \exists\ n \in N \mid S_n = 0) = 1$

b) $P(S_n = 0\ infinitely\ often) = 1$

c) $\sum_{n=0}^{\infty} P(S_n = 0) = \infty$

A ***transient walk*** on the other hand, is a simple random walk that is not recurrent, i.e., the random walker has a probability less than one of returning to the origin.

3 Z_1 Case

Theorem. *The simple random walk on $\mathbb{Z}^1$ is recurrent.*

Assume that the walker starts at the origin $x = 0$. Let S_n be the walker's location at step n, and X_i be a random variable representing the i-th step with $0 < i \leq n$. We let each step to the right to be represented by the value 1, and each step to the left to be represented by the value -1. By the definition of random walk, the walker has equal probability of taking a step in any direction:

$$P(X_i = +1) = P(X_i = -1) = \tfrac{1}{2d} = \tfrac{1}{2}$$

Hence, the position of the walker after n steps is given by the summation of the values for each individual step.

$$S_n = X_1 + X_2 + ... + X_n = \textstyle\sum_{i=1}^{n} X_i$$

To demonstrate recurrence, our goal is to determine the likelihood of the walker arriving back at the starting point after n steps. Therefore, we define u_n as the probability of the walker being at the origin following n steps.

$$u_n = P(S_n = 0) = \textbf{?}$$

However, in order for the walker to return to the origin, for each step taken in one direction, there must be a corresponding step taken in the opposite direction to cancel it out. Only when the number of left turns is equal to the number of right turns will the walker be able of returning to the starting point. Whether the number of left steps and the number of right steps are even or odd, the summation of them will be even. Hence, for our purpose, we can focus only in walks with an even number of steps:

$$u_{2n} = P(S_{2n} = 0) = \textbf{?}$$

Assuming the walker completes a $2n$ step walk in $\mathbb{Z}^1$, in each step, they can either move to the right (+1) or move to the left (-1). Since there are 2 choices at each of the 2n steps, by the multiplication principle, there are 2^{2n} different paths the walker can take.

To get an equal number of left steps and right steps, out of the total $2n$ steps, we need to choose n steps to be "left" steps. The remaining steps will automatically be "right" steps. The problem reduces to selecting n "left" steps out of a total of $2n$ steps. Hence, There are $\binom{2n}{n}$ ways to select n steps from a total of $2n$ steps. So, the probability of returning to the origin after $2n$ steps is:

$$\begin{aligned} u_{2n} = P(S_{2n} = 0) &= \frac{\binom{2n}{n}}{2^{2n}} \\ &= \frac{(2n)!}{n!(2n-n)!}\frac{1}{2^{2n}} \\ &= \frac{(2n)!}{(n!)^2\ 2^{2n}} \end{aligned} \tag{1}$$

Here, n is an infinitely large number, making the calculation of the above probability challenging. To solve this problem, we can use Stirling's Approximation as a tool.

Stirling's Approximation is a formula used to approximate the factorial of a large number that comes from comparing the factorial (a discrete function) to a continuous function called the gamma function. By approximating the discrete function with a continuous one, it becomes easier to analyze and compute.

Stirling's Approximation. *As* $n \to \infty$, $n! \approx \sqrt{2\pi n}e^{-n}n^n$

The exponential part of the formula comes from the natural growth behavior of factorials, which is similar to exponential functions. Using this exponential relationship helps to simplify the approximation.

The $\sqrt{2\pi n}$ part of the formula might seem arbitrary, but it's included to make the approximation more accurate. It comes from a mathematical technique called asymptotic analysis, which studies how functions behave as their inputs get very large. Substituting the formula to our result, as n approaches infinity, it provides an increasingly accurate estimate of n!:

$$\begin{aligned} \therefore\ u_{2n} &\approx \frac{(2n)^{2n}e^{-2n}\sqrt{4\pi n}}{n^{2n}e^{-2n}\ 2\pi n\ 2^{2n}} \\ &= \frac{\sqrt{4\pi n}}{2\pi n} \\ &= \frac{1}{\sqrt{\pi n}} \end{aligned} \tag{2}$$

By definition of recurrent walk, if the walker returns to the origin infinitely many times, then the walk is recurrent. To find out how many times the walker returns, let's define a random variable J_{2n}, which takes the value of 1 if the walker returns to the origin at step $2n$ and 0 otherwise. Intuitively, the expected value for our random variable will be $\frac{1}{\sqrt{\pi n}}$:

$$\mathbb{E}[J_{2n}] = 1 \cdot P(J_{2n} = 1) + 0 \cdot P(J_{2n} = 0) = \frac{1}{\sqrt{\pi n}} \cdot 1 + (1 - \frac{1}{\sqrt{\pi n}}) \cdot 0 = \frac{1}{\sqrt{\pi n}} \tag{3}$$

Since J_{2n} yields a value of 1 for each return and 0 otherwise, the total number of returns is $\sum_{n=1}^{\infty} J_{2n}$. Therefore, the expected number of returns can be calculated as below:

$$\begin{aligned} \mathbb{E}[\sum_{n=1}^{\infty} J_{2n}\ ] = \sum_{n=1}^{\infty} \mathbb{E}[J_{2n}] &= \sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} \\ &= \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \end{aligned} \tag{4}$$

Note that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a p-series with $p = \frac{1}{2}$. Since $0 < p \leq 1$, the p-series diverges. Since the original series is just the p-series scaled by a constant factor, the entire series diverges as well. This means that the sum of its terms does not approach a finite value as the number of terms approaches infinity.

$$\sum_{n=1}^{\infty} J_{2n} = \infty$$
$$\therefore \sum_{n=0}^{\infty} P(S_{2n} = 0) = \infty \tag{5}$$

Therefore, the simple random walk on $\mathbb{Z}^1$ is recurrent.

4 Z_2 Case

Theorem. *The simple random walk on $\mathbb{Z}^2$ is recurrent.*

$\mathbb{Z}^2$ can be thought as a lattice is a rectangular grid with equally spaced horizontal and vertical lines, forming intersections called lattice points.

For each step, the walker can either move along the x-axis (left or right) or the y-axis (up or down). The probability of moving in the x-axis is not affected by the walker's position or movements in the y-axis, and vice versa. Hence, the decision to move in the x or y direction is independent of the walker's previous movements, which means that the x-axis and y-axis movements can be considered as two separate one dimensional random walks.

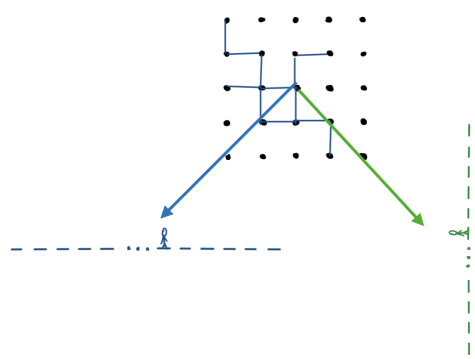

Figure 2

For two independent events, the probability of both events occurring simultaneously is equal to the product of their individual probabilities. This trick allows us to find the probability of the walker returning into the origin after $2n$ steps in $\mathbb{Z}^2$ by multiplying the probabilities of returning to the origin in each of the one-dimensional random walks along the x and y axes.

$$P(S_{2n} = 0) = \frac{1}{\sqrt{\pi n}} \frac{1}{\sqrt{\pi n}} = \frac{1}{\pi n} \tag{6}$$

Therefore, the expected number of returns in $\mathbb{Z}_2$ is

$$\begin{aligned}\mathbb{E}[\sum_{n=1}^{\infty} J_{2n}\ ] = \sum_{n=1}^{\infty} \mathbb{E}[J_{2n}] &= \sum_{n=1}^{\infty} \frac{1}{\pi n} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \\ &= \infty \end{aligned} \tag{7}$$

If not convinced, we can also follow the direct counting proof approach as we did in the previous proof to arrive at the same conclusion. Since now there are 4 choices at each of the $2n$ steps, applying the multiplication principle, there are 4^{2n} different paths the walker can take.

To be able to return to the origin, we need to choose the number of paths with k steps left, k steps right, $n-k$ steps up, $n-k$ steps down over the total number of $2n$ steps. Hence, There are $\binom{2n}{k,k,n-k,n-k}$ ways to select n steps from a total of $2n$ steps. So, the probability of returning to the origin after $2n$ steps is:

$$\begin{aligned} u_{2n} = P(S_{2n} = 0) &= \frac{\binom{2n}{k,k,n-k,n-k}}{4^{2n}} \\ &= \frac{1}{4^{2n}} \frac{(2n)!}{k!k!(n-k)!(n-k)!} \\ &= \frac{1}{4^{2n}} \sum_{k=0}^{n} \frac{(2n)!n!n!}{k!k!(n-k)!(n-k)!n!n!} \\ &= \frac{1}{4^{2n}} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \\ &= \frac{1}{4}^{2n} \binom{2n}{n} \binom{2n}{n} \end{aligned} \tag{8}$$

So we get $u_{2n} = \left(\frac{1}{2^{2n}}\binom{2n}{n}\right)^2$ which is just the square of the result from $\mathbb{Z}^1$. In this case, by substituting the result from our first proof we get:

$$u_{2n} = (\frac{1}{\sqrt{\pi n}})^2 = \frac{1}{\pi n} \tag{9}$$

Hence,

$$\begin{aligned}\mathbb{E}[\sum_{n=1}^{\infty} J_{2n}\ ] = \sum_{n=1}^{\infty} \mathbb{E}[J_{2n}] &= \sum_{n=1}^{\infty} \frac{1}{\pi n} \\ &= \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \end{aligned} \tag{10}$$

The harmonic series is well-known for being a divergent series, meaning that as you take the sum over an increasing number of terms, the value of the sum grows without bound.

This scaling factor $\frac{1}{\pi}$ does not change the fact that the sum of the terms $\frac{1}{n}$ grows without bound. The resulting series is still a divergent series, only with a different rate of growth due to the scaling factor. Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} J_{2n} &= \infty \\ \therefore \sum_{n=0}^{\infty} P(S_{2n} = 0) &= \infty \end{aligned} \tag{11}$$

Therefore, the simple random walk on $\mathbb{Z}^2$ is recurrent.

5 Z_3 Case

Theorem. *The simple random walk on $\mathbb{Z}^3$ is transient.*

The third dimension is where the scenario changes. Based on the transient definition, we now have to prove that there is a possibility of the walker never returning to the origin.

We will calculate u_{2n} for random walks on $\mathbb{Z}^3$. Given that there are three axes, each with a pair of positive and negative directions, we have a total of 6 distinct directions to move in.

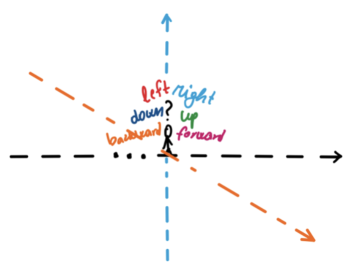

Figure 3

Thus, for $2n$ steps, there are 6^{2n} possible random walks. Among these walks, our focus is on those that lead back to the origin.

We let j be the number of moves along the x-axis, and k be the number of moves along the y-axis. To return to the origin after completing $2n$ steps, we need an even count of moves for each axis, with an equal number of positive and negative moves for every axis. Therefore, there are n - j - k steps along the z-axis. The remaining n from $2n$ moves will be in the opposite directions for each respective axis.

In other words, the number of possible paths of length $2n$ that end at the origin of $\mathbb{Z}^3$, is represented by choosing the number of paths with j steps left, j steps right, k steps up, k steps down, $n - j - k$ steps forward, $n - j -$

k steps backward, over the total number of steps $2n$. Therefore, there are $\binom{2n}{j,j,k,k,n-j-k,n-j-k}$ possible paths that end at the origin. Extending the idea from the previous proofs, we get:

$$\begin{aligned} u_{2n} = P(S_{2n} = 0) &= \frac{\binom{2n}{j,j,k,k,n-j-k,n-j-k}}{6^{2n}} \\ &= \frac{1}{6^{2n}} \sum_{\substack{j,k>0 \\ j+k\leq n}} \frac{(2n)!}{j!j!k!k!(n-j-k)!(n-j-k)!} \\ &= (\frac{1}{6})^{2n} \binom{2n}{n} \sum_{\substack{j,k>0 \\ j+k\leq n}} (\frac{n!}{j!k!(n-j-k)!})^2 \\ &= (\frac{1}{2})^{2n} \binom{2n}{n} \sum_{\substack{j,k>0 \\ j+k\leq n}} ((\frac{1}{3})^n \frac{n!}{j!k!(n-j-k)!})^2 \end{aligned} \tag{12}$$

What is left now is to asymptotically estimate the above probability. The probability of placing n objects in 3 places is maximized when j, k and $n-j-k$ are all as close to $\frac{n}{3}$ as possible. Since, we are mainly interested in the asymptotics of the value, for ease of computation we assume that $\frac{n}{3}$ is an integer. Hence,

$$(\frac{1}{3})^n (\frac{n!}{j!k!(n-j-k)!}) \leq \frac{n!}{3^n \lfloor\frac{n}{3}\rfloor! \lfloor\frac{n}{3}\rfloor! \lfloor\frac{n}{3}\rfloor!} \tag{13}$$

Following the same thought process, we get

$$\sum_{j,k} \frac{n!}{3^n j!k!(n-j-k)!} = 1 \tag{14}$$

since, there is only one way of choosing all objects from the total number of objects. In this way, we can observe that

$$u_{2n} \leq (\frac{1}{2})^{2n} \binom{2n}{n} \frac{n!}{3^n \lfloor\frac{n}{3}\rfloor! \lfloor\frac{n}{3}\rfloor! \lfloor\frac{n}{3}\rfloor!} = (\frac{1}{2})^{2n} \binom{2n}{n} \frac{n!}{3^n (\lfloor\frac{n}{3}\rfloor!)^3} \tag{15}$$

Applying Stirling's Approximation, we get that

$$u_{2n} \leq (\frac{1}{2})^{2n} \binom{2n}{n} \frac{c}{n} \approx \frac{1}{\sqrt{\pi n}} \frac{c}{n} = \frac{c}{n^{\frac{3}{2}}} \tag{16}$$

where c is probably $3\sqrt{3}$, but for our purpose we treat it as some positive constant. Defining the random variable J_{2n} as in the previous proofs, we get:

$$\begin{aligned} \mathbb{E}[\sum_{n=1}^{\infty} J_{2n}\ ] = \sum_{n=1}^{\infty} \mathbb{E}[J_{2n}] &= \sum_{n=1}^{\infty} \frac{c}{n^{\frac{3}{2}}} \\ &= c \sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} < \infty \end{aligned} \tag{17}$$

The sum on the left of the inequality converges as a p-series with $p > 1$.

$$\therefore \sum_{n=0}^{\infty} P(S_{2n} = 0) < \infty \tag{18}$$

Therefore, the simple random walk on $\mathbb{Z}^3$ is transient.

The same ideas presented for $\mathbb{Z}^3$ can be extended for $d > 3$ and in doing so we see that for $d \geq 3$ the simple random walk is transient. However, proofs of higher dimensions are behind the scope of this paper.

6 Summary

Although no proofs for higher dimensions are discussed in this paper, the following table presents an analysis of random walks across varying dimensions, focusing on the probability of a walker returning to the origin. In lower dimensional lattice paths, such as $\mathbb{Z}^1$ and $\mathbb{Z}^2$, the walker is guaranteed to return to the origin with a probability of 1, exhibiting recurrent behavior. Counter intuitively, in $\mathbb{Z}^3$ the walker has only 34% chance of returning to the origin. As the dimensions increase, the probability of returning to the origin significantly decreases, indicating that higher-dimensional walks tend to be transient in nature.

d	u_{2n}
1	1
2	1
3	0.3405
4	0.1932
5	0.1351
6	0.1047
7	0.0858
8	0.0729

Figure 4

7 Conclusion

In this paper, we have explored Polya's random walk theorem, which states that a simple random walk on a d-dimensional lattice path is recurrent for dimensions $d = 1, 2$ and transient for dimensions $d \geq 3$. We have presented a detailed discussion on the direct counting proof for this theorem and examined the recurrence properties of random walks in Z_1 and Z_2, contrasting them with the transience property observed in Z_3.

For the one-dimensional case, we demonstrated recurrence by computing the probability of the walker returning to the origin after n steps. In the two-dimensional case, we proved the recurrence by considering the two axes as independent one-dimensional random walks. In contrast, for the three-dimensional case, we established the transience property by computing the probability of returning to the origin and demonstrating that the sum of probabilities for returning after an even number of steps converges to a finite value.

Our focus was only on simple symmetric random walks. Investigating the behavior of random walks with different step size distributions, varying lattice structures, or incorporating external forces to model more complex systems would be the next step for further exploration of Polya's Random Walk Theorem. Additionally, exploring random walks in higher dimensions and their implications in areas such as data analysis, network routing, and population dynamics can be a promising direction for advancing the applications of this theorem.

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