Risk Sharing in Insurance*

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1 Introduction

Traditional insurance is based on a centralized model. Each insurance policy has a bilateral contractual relationship between a policyholder and an insurer. Decentralized models allow a pool of participants to spread out risks among each other, removing the role of central authority who traditionally takes on all the risks. Peer-to-peer (P2P) insurance model is based on a social network concept where a group pool their resources to compensate each other for losses and cut down on the cost of insurance. Companies such Friendsurance and Lemonade are known to offer P2P insurance products.

Risk sharing has emerged as a vital mechanism in financial markets to address risks that are often too large or unpredictable for conventional insurance to handle. For example, the Caribbean Catastrophe Risk Insurance Facility (CCRIF) was created in response to the significant economic devastation caused by natural disasters such as hurricanes and earthquakes in the Caribbean region. The severe impact of Hurricane Ivan in 2004, which led to over $6 billion in losses across nine Caribbean countries, highlighted the urgent need for innovative risk management solutions, particularly for developing nations with limited financial resources. Recognizing this need, the World Bank and 19 Caribbean and Central American countries established the CCRIF in 2007 as a sovereign trust fund for disaster financing. This arrangement allows member countries to collectively pool their risks, diversify their portfolios, retain a portion of the risk through joint capital, and transfer excess exposure to global reinsurance and capital markets. Although these countries are geographically close, they are not always simultaneously affected by disasters, enabling them to share the financial impact and support one another when catastrophes occur. The success of catastrophe risk pooling has led to its adoption in other regions, including the Florida Hurricane Catastrophe Fund (U.S.), Flood Re (U.K.), African Risk Capacity (Bermuda), and the Pacific Catastrophe Risk Assessment and Financing Initiative (Pacific Islands). For a comprehensive analysis of catastrophe risk pooling.

The study of risk sharing arises naturally from insurance, particularly from the reinsurance line of business. In the past few years, there has been a surge in risk sharing research related to P2P insurance models. Feng et al. [2023] introduce a hierarchical P2P model with applications to flood insurance products. An extension to the dynamic, time-dependent setting is done by Abdikerimova et al. [2024]. Denuit et al. [2022] lay basic foundational properties that are deemed attractive/desirable in a risk sharing model. A specific example of a rule that is based on quantile is studied in Dhaene et al. [2025].

2 Mathematical preliminaries and classical model

2.1 Mathematical formulation of risk sharing

To formulate the risk sharing problem, we first introduce some notation.

Definition 2.1.1. *Let $X = (X_1, \ldots, X_n)^T$ be a non-negative random vector with entries X_i representing the loss of agent i. A risk sharing rule is a nonnegative function $h : \mathbb{R}^n \to \mathbb{R}^n$. We say that h satisfies the* ***full-allocation property*** *(zero-balance conservation) if*

$$\sum_{i=1}^{n} h_i(X) = \sum_{i=1}^{n} X_i. \tag{1}$$

Remark 2.1.2. In the classical risk sharing literature, losses are aggregated first before redistributed to individual agents, so h_i is a function for $\sum_{i=1}^n X_i$. On the other hand, the formulation of P2P models relies on the source of the loss, i.e. h_i is a function of $(X_1, \ldots, X_n)^T$. Our research focuses on the later type.

We will now introduce some desirable properties of risk-sharing models. An important property is called **actuarial fairness**, which say that, on average, the loss of agent i before risk sharing is the same as the loss after risk sharing, that is,

$$\mathbb{E}[h_i(X)] = \mathbb{E}[X_i]. \tag{2}$$

In other words, the expected net return of this risk sharing scheme must be zero. Another property is called the **Pareto optimality condition**, which is closely related to some concepts in economics. This property asserts that the post-exchange variance must be minimized. Mathematically, this leads to the following optimization problem

$$\min \sum_{i=1}^n \mathrm{Var}(h_i(X)).$$

2.2 n-agent model and results

An example of a risk sharing model is the linear (pro rata) rule:

$$h(X) = AX,$$

where A is any matrix that has columns that sum to 1, that is, $\mathbf{1}^T A = \mathbf{1}^T$. We will refer to the matrix $A = (\alpha_{ij})_{i,j=1}^n$ as the **allocation matrix**.

Example 2.2.1. When $n = 2$, the linear risk sharing rule reduces to

$$H(X) = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & 1-\alpha_{22} \\ 1-\alpha_{11} & \alpha_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}.$$

This formulation is easily generalized to the n-agent problem. Let $\tilde{X} = (\tilde{X}_1, \ldots, \tilde{X}_n)$ denote the pre-exchange losses of all agents with mean $\mathbb{E}[\tilde{X}] = \mu$ and positive definite covariance $\mathrm{Cov}(\tilde{X}) = \Sigma$. The components of Σ are denoted $\sigma_{ij} = \mathrm{Cov}(\tilde{X}_i, \tilde{X}_j) = \rho_{ij}\sigma_i\sigma_j$ where $\sigma_i = \mathrm{std}(\tilde{X}_i)$ and $\rho_{ij} = \mathrm{corr}(\tilde{X}_i, \tilde{X}_j)$. Consider the allocation matrix A and suppose that after the group enters a risk-sharing agreement, the new joint risk is given by $X = A\tilde{X}$. We use the notation $\mathbf{1} = [1, \ldots, 1]^T \in \mathbb{R}^n$. The allocation ratio matrix of the optimal risk sharing pool will be the minimizer of the sum of the variances of the individual agents subject to actuarial fairness and loss conservation conditions. Mathematically, the following theorem was formulated in Feng et al. [2023]:

Theorem 2.2.2 (Feng et al., 2023)**.** *Consider the problem of finding the optimal allocation matrix $\tilde{A}$ that minimizes*

$$\begin{aligned} \min_{A \in \mathbb{R}^{n\times n}} \sum_{i=1}^n \mathrm{Var}(X_i) &= \min_{A \in \mathbb{R}^{n\times n}} \mathrm{tr}(A\Sigma A^T) \\ \textit{s.t.} \quad A\mu &= \mu, \\ \mathbf{1}^T A &= \mathbf{1}^T \end{aligned} \tag{3}$$

Then,

$$\tilde{A} = \frac{1}{n}\mathbf{1}\mathbf{1}^T + \frac{1}{\mu^T \Sigma^{-1}\mu}\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mu\mu^T\Sigma^{-1}. \tag{4}$$

Proof. We use the method of Lagrange multipliers. Define the Lagrangian,

$$\mathcal{L} = \mathrm{tr}(A\Sigma A^T) + (\mathbf{1}^T - \mathbf{1}^T A)\lambda + \beta^T(\mu - A\mu).$$

Differentiating, we have that

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial A} = A\Sigma^T + A\Sigma - \mathbf{1}\lambda^T - \beta\mu^T, \\ 0 &= \frac{\partial \mathcal{L}}{\partial \lambda} = \mathbf{1}^T - \mathbf{1}^T A, \\ 0 &= \frac{\partial \mathcal{L}}{\partial \beta} = \mu - A\mu, \end{aligned}$$

and so the minimizer $\tilde{A}$ satisfies

$$\tilde{A} = (\mathbf{1}\lambda^T + \beta\mu^T)(2\Sigma)^{-1}.$$

Inserting into the second equation, we have

$$\begin{aligned}
0 &= \mathbf{1}^T - \mathbf{1}^T(\mathbf{1}\lambda^T + \beta\mu^T)(2\Sigma)^{-1} \\
\implies \mathbf{1}^T &= \mathbf{1}^T(\mathbf{1}\lambda^T + \beta\mu^T)(2\Sigma)^{-1} \\
\implies \mathbf{1}^T(2\Sigma) &= \mathbf{1}^T(\mathbf{1}\lambda^T + \beta\mu^T) \\
\implies \mathbf{1}^T\mathbf{1}\lambda^T + \mathbf{1}^T\beta\mu^T &= \mathbf{1}^T(2\Sigma) \\
\implies \mathbf{1}^T\mathbf{1}\lambda^T &= \mathbf{1}^T(2\Sigma - \beta\mu^T) \\
\implies \lambda^T &= \frac{1}{n}\mathbf{1}^T(2\Sigma - \beta\mu^T).
\end{aligned}$$

Inserting into the third equation, we have

$$\begin{aligned}
0 &= \mu - (\mathbf{1}\lambda^T + \beta\mu^T)(2\Sigma)^{-1}\mu \\
\implies 0 &= \mu - \mathbf{1}\lambda^T(2\Sigma)^{-1}\mu - \beta\mu^T(2\Sigma)^{-1}\mu \\
\implies 0 &= \mu - \frac{1}{n}\mathbf{1}\mathbf{1}^T(2\Sigma - \beta\mu^T)(2\Sigma)^{-1}\mu - \beta\mu^T(2\Sigma)^{-1}\mu \\
\implies 0 &= \mu - \frac{1}{n}\mathbf{1}\mathbf{1}^T\mu + \frac{1}{n}\mathbf{1}\mathbf{1}^T\beta\mu^T(2\Sigma)^{-1}\mu - \beta\mu^T(2\Sigma)^{-1}\mu \\
\implies 0 &= \left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mu - \left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\beta\mu^T(2\Sigma)^{-1}\mu \\
\implies 0 &= \left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)(\mu - \beta\mu^T(2\Sigma)^{-1}\mu) \\
\implies \beta &= (\mu^T(2\Sigma)^{-1}\mu)^{-1}\mu.
\end{aligned}$$

To summarize, we have that

$$\begin{aligned}
\tilde{A} &= (\mathbf{1}\lambda^T + \beta\mu^T)(2\Sigma)^{-1}, \\
\lambda^T &= \frac{1}{n}\mathbf{1}^T(2\Sigma - \beta\mu^T), \\
\beta &= (\mu^T(2\Sigma)^{-1}\mu)^{-1}\mu,
\end{aligned}$$

and so

$$\begin{aligned}
\tilde{A} &= \left(\frac{1}{n}\mathbf{1}\mathbf{1}^T(2\Sigma - (\mu^T(2\Sigma)^{-1}\mu)^{-1}\mu\mu^T) + (\mu^T(2\Sigma)^{-1}\mu)^{-1}\mu\mu^T\right)(2\Sigma)^{-1} \\
&= \frac{1}{n}\mathbf{1}\mathbf{1}^T(2\Sigma)(2\Sigma)^{-1} - \frac{1}{n}\mathbf{1}\mathbf{1}^T(\mu^T(2\Sigma)^{-1}\mu)^{-1}\mu\mu^T(2\Sigma)^{-1} + (\mu^T(2\Sigma)^{-1}\mu)^{-1}\mu\mu^T(2\Sigma)^{-1} \\
&= \frac{1}{n}\mathbf{1}\mathbf{1}^T + (\mu^T(2\Sigma)^{-1}\mu)^{-1}\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mu\mu^T(2\Sigma)^{-1} \\
&= \frac{1}{n}\mathbf{1}\mathbf{1}^T + \frac{1}{\mu^T\Sigma^{-1}\mu}\left(I - \frac{1}{n}\mathbf{1}\mathbf{1}^T\right)\mu\mu^T\Sigma^{-1}
\end{aligned}$$

as desired. □

Remark 2.2.3. In the special case, $\mu_1 = \mu_2 = \cdots = \mu_n$, then $\tilde{A} = \frac{1}{n}I$. That is, the optimal allocation is to split all losses equally among all agent.

Remark 2.2.4. It is often useful to rewrite the objective function via Cholesky decomposition for numerical optimization purposes. Indeed, since Σ is positive definite, we have that $\Sigma = LL^T$ for some lower triangular matrix L and therefore

$$\operatorname{tr}(A\Sigma A^T) = \operatorname{tr}(ALL^TA^T) = \operatorname{tr}((AL)(AL)^T) = (AL)^T(AL) = \|AL\|_F^2$$

where $\|\cdot\|_F$ is the Frobenius norm.

Remark 2.2.5. Each agent has post-pooling variance

$$\operatorname{Var}(X_i) = \frac{1}{n^2}\mathbf{1}^T\Sigma\mathbf{1} + \frac{\mu_i^2}{\mu^T\Sigma^{-1}\mu} - \frac{1}{n^2\mu^T\Sigma^{-1}\mu}\left(\sum_{i=1}^n \mu_i\right)^2 \geq \frac{1}{n^2}\mathbf{1}^T\Sigma\mathbf{1}.$$

This follows from noting that $(\mu^T\Sigma^{-1}\mu)^{-1} > 0$ since Σ is positive definite and applying the Cauchy-Schwarz inequality on μ:

$$\sum_{i=1}^n \mu_i^2 \geq \frac{1}{n^2}\left(\sum_{i=1}^n \mu_i\right)^2.$$

This implies that the post-pooling variance of each agent is not less than the mean covariance of all agents

2.3 Python implementation

To investigate how changing the difference in the mean expected loss between two agents in this simplified example we perform a numerical experiment to implement this optimization problem for a series of different mean loss vectors. All codes used throughout this paper can be found at: `https://github.com/Harbour-N/Risk_sharing_GSMMC/tree/main`. We generate 1,000 random losses for each agent (X_1 and X_2) and perform the optimization to find the optimal coefficients of the allocation matrix A. We compare both a numerical optimization scheme (differential evolution) with the analytical solution provided in Feng et al. [2023]. We then repeat this keeping the mean loss of Agent One the same but multiplying Agent Two's mean by a scale factor. The results are shown in Figure 1.

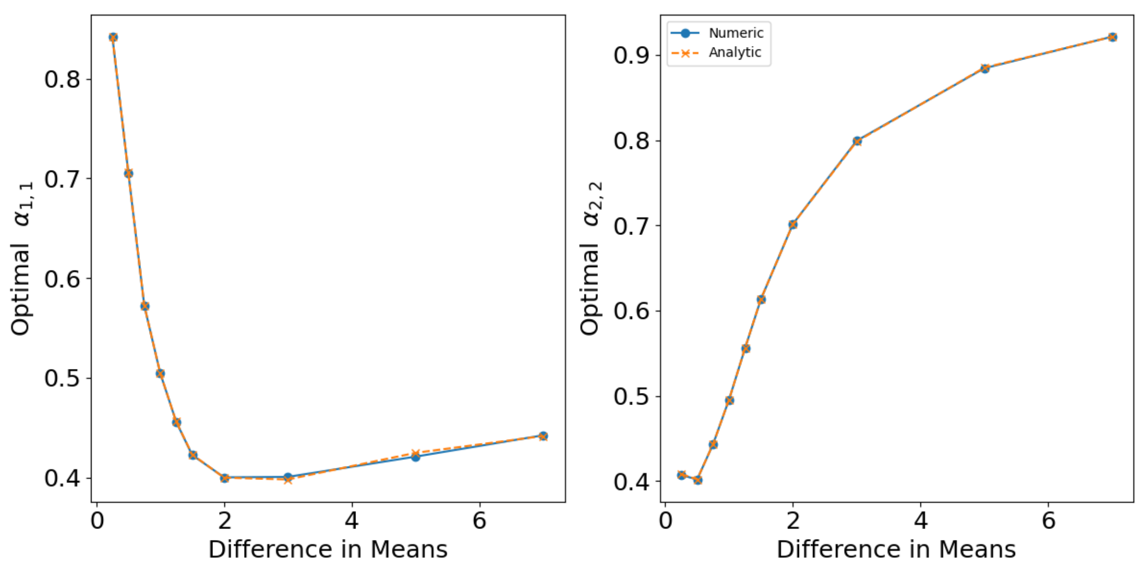


Figure 1: Optimal $\alpha_{1,1}$ and $\alpha_{2,2}$ for a series of different losses. The x-axis represents the difference(ratio) in means between the two agents loss samples.

Agent One has a fixed mean of 0.1, while the mean loss of Agent Two is varied. We define the difference in means in the multiplicative sense, that is:

$$\text{difference in means} = \frac{\bar{X}_2}{\bar{X}_1}. \tag{5}$$

Thus, a difference in mean of 0.5 corresponds to $\bar{X}_2$ is half $\bar{X}_1$. The means are the same when the difference in means is 1. Figure 1 verifies that when the means are the same this corresponds to $\alpha_{1,1} = 0.5$ and $\alpha_{2,2} = 0.5$, as expected – the agents share risk equally.

When the difference in means is small then the mean expected loss of Agent One is much higher than Agent Two thus they are allocated much more of the risk. As the mean loss of Agent Two increases relative to Agent One its allocation decreases.

3 Multi-period extension

A multi-period peer-to-peer (P2P) risk sharing model refers to a framework in which a group of agents (participants) repeatedly share risk among themselves over multiple time periods. This concept is a generalization of the classical static risk-sharing model (which occurs at a single point in time) by introducing a dynamic, temporal structure where sharing rules, wealth, and risk exposure evolve over time. Most of the current literature focuses on single-period P2P risk-sharing models. Agents can benefit from participating in long-term risk sharing, enabling them to smooth out fluctuations arising from risks and uncertainties over time. The literature on dynamic risk sharing has primarily focused on the objective of the temporal effect on the participants' long-term capital.

3.1 Spatial versus Temporal Effect

In a single-period peer-to-peer (P2P) risk-sharing model, such as catastrophe risk pooling, the system operates on a "pay-as-you-go" basis, where losses are distributed and settled immediately as they arise, rather than being managed in advance or deferred to the future. This structure inherently avoids any transfer of risk between generations, preventing situations where the younger cohort bears the financial burden of the older participants. Consequently, the primary function of such a system is spatial risk diversification, ensuring that the financial impact of individual losses is shared collectively among all members.

In this study, we extend the analysis to a multi-period P2P risk-sharing framework under a similar "pay-as-you-go" approach. However, unlike the static model, the multi-period setting allows for the balance of risk-sharing to be carried forward across different periods. This introduces a temporal dimension to the risk allocation, enabling partial smoothing of losses over time. With multiple periods under consideration, it becomes crucial to develop mechanisms that not only distribute risk among participants but also enhance financial stability throughout the duration of the risk-sharing arrangement.

In a P2P risk-sharing scheme in place, the participants can trade losses with each other to provide some level of temporary relief. The participants split the losses among themselves, which in turn controls the temporary effect on the participants' long-term capital, which stabilizes the reserve fund.

3.2 Problem Formulation

Recall that for a single-period problem, we have the following two-agent problem:

$$h(X) = AX = \begin{bmatrix} \alpha_{11} & 1-\alpha_{22} \\ 1-\alpha_{11} & \alpha_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}. \tag{6}$$

Considering a multi-period problem, the coefficients of the A matrix are time-dependent, i.e, $A := A(t)$, $\alpha_{11} := \alpha_{11}(t)$ and $\alpha_{22} := \alpha_{22}(t)$, as well as the losses $X_1 := X_1(t)$ and $X_2 = X_2(t)$. To get the optimal values of the coefficients of the matrix A_t, we solve the optimization problem in terms of all the time dependent matrices. For example, given time steps $t = 0, 1, \dots, T$, the optimization problem is formulated as

$$\min_{\{A(t)\}_{t=0}^{T-1}} \mathbb{E}[\mathbf{f}(X(t))], \tag{7}$$

for some function $\mathbf{f}(X(t))$ of the losses. As discussed in Abdikerimova et al. [2024], one could use the optimization problem in Equation (7) to maximize the expected countries' reserve amounts $R(T)$ at some final time T. In particular, for all $t = 1, \dots, T$, we define $R(t)$ via the recursive relation

$$R(t) := R(t-1) - A(t-1)X(t). \tag{8}$$

Hence, the optimization problem in Equation 7 can be stated as:

$$\max_{\{A(t)\}_{t=0}^{T-1}} \mathrm{Tr}(\lambda \mathbb{E}[R^{\mathrm{T}}(T)] - \mathbf{\Omega}\mathbb{E}[R(T)R^{\mathrm{T}}(T)]), \tag{9}$$

with the weight vector λ and the weight diagonal matrix $\mathbf{\Omega}$. These weights can be though of a balancing two potentially competing objectives – maximizing the final reserve, while minimizing the variation in the final reserve, subject to the constraints,

$$\mathbf{1}^{\mathrm{T}} A(t) - \mathbf{1}^{\mathrm{T}} = 0 \qquad \text{(zero-balance loss conservation)}. \tag{10}$$

The aim of the optimization problem is to find the optimal values of the coefficients of the matrices $A(t)$ to maximize the expected value of terminal reserves and minimize their variations.

3.3 Implementation

Extending on Section 2.3, we implement the following problem setup, inspired by Feng et al. [2023]. Three participants participate in an optimal dynamic risk-sharing allocation for 7 periods. We assume that losses $(X_1, \ldots, X_{\mathrm{T}})$ are independent and identically distributed over time. They follow a multivariate normal distribution with the vector of expectations = $[55, 70, 100]$ and the variance-covariance matrix given by

$$\Sigma = \begin{bmatrix} 225 & 50 & 20 \\ 50 & 121 & 30 \\ 20 & 30 & 625 \end{bmatrix}.$$

The weights are pre-set as follows

$$\Omega = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{and} \quad \lambda' = [1,\ 1/2,\ 2].$$

All participant are set to have the same initial reserve $R_0 = [1500, 1500, 1500]$. We use Monte Carlo simulations to generate 1000 sample paths of the realized losses over 7 periods. For each sample path, we calculate the allocation matrix A_t given by solving the optimization problem 7. The numerical results are presented in Figures 2 and 3.

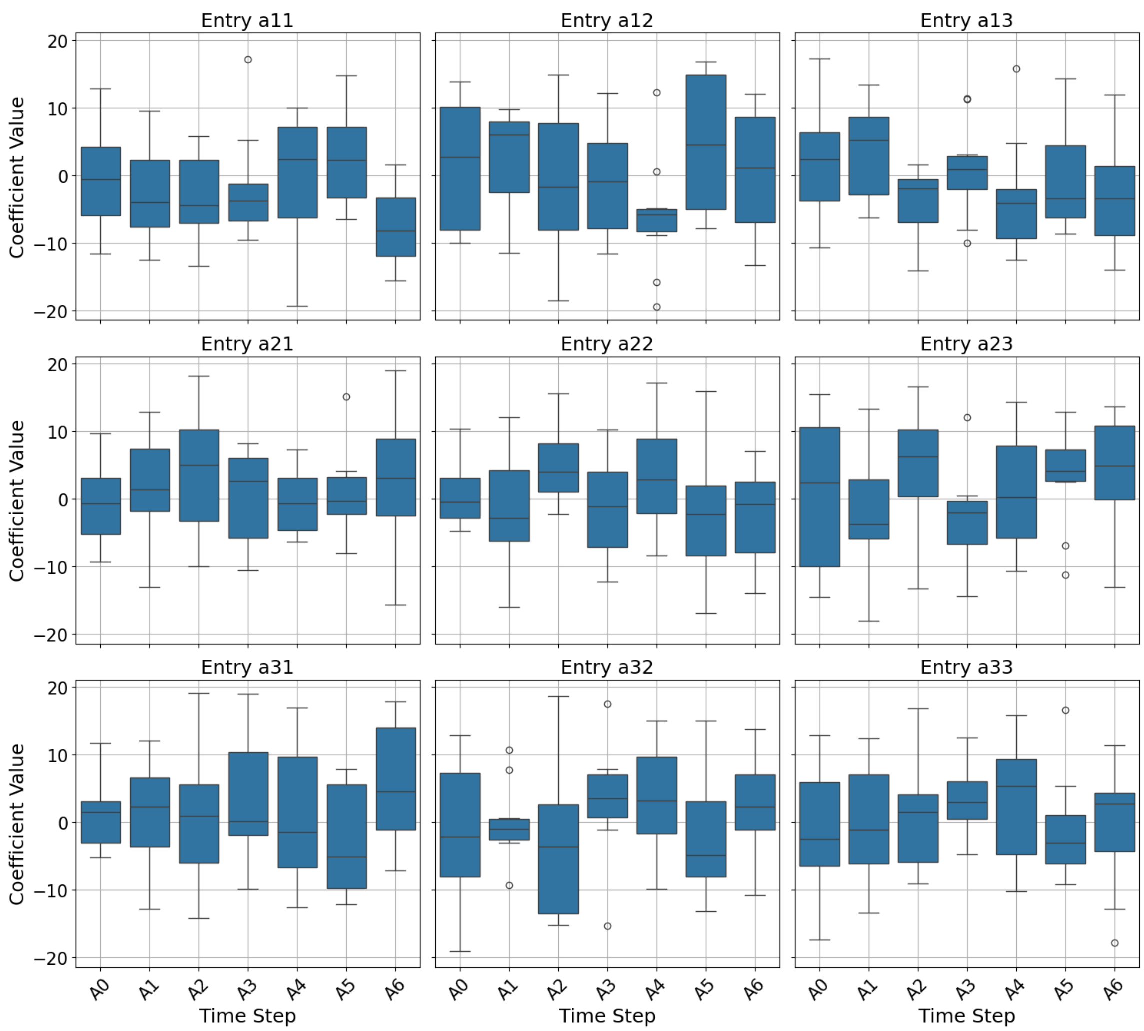


Figure 2: Example coefficients over time

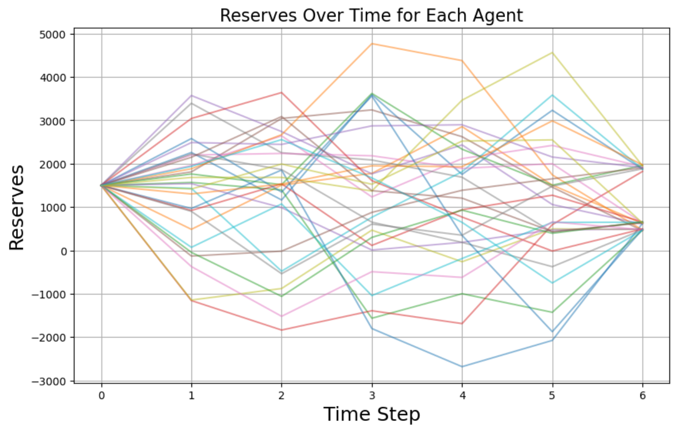


Figure 3: Reserve over time for the agents following the allocation matrices given in Figure 2

4 New risk sharing rules

In this section, we investigate some risk sharing rules beyond the linear model.

4.1 Ordinary deductible

4.1.1 Two Agents

We consider a risk-sharing mechanism between two agents who each faces uncertain losses, denoted by the random variables, X_1 and X_2, respectively. To balance fairness and individual responsibility, we introduce a deductible-based redistribution rule. Each agent retains responsibility for their own losses up to a personal deductible threshold d_i, while any excess loss beyond that threshold is absorbed by the other agent. Formally, we observe that

$$X_i = \underbrace{\min(X_i, d_i)}_{\text{self}} + \underbrace{(X_i - d_i)^+}_{\text{distribute}}, \quad i = 1, 2 \tag{11}$$

where $(X_i - d_i)^+ = \max(X_i - d_i, 0)$, the ReLU function. We define the corresponding risk sharing rule

$$h_i(X) = \min(X_i, d_i) + (X_j - d_j)^+, \quad i, j = 1, 2;\ i \neq j. \tag{12}$$

Clearly,

$$h_1(X) + h_2(X) = X_1 + X_2$$

so the rule satisfies zero-balance conservation. We wish to impose actuarial fairness and Pareto optimality conditions. That is, we want to solve the optimization problem

$$\begin{aligned} \min_{d_1, d_2}\ & \operatorname{Var}(h_1(X)) + \operatorname{Var}(h_2(X)) \\ \text{s.t.}\ & \mathbb{E}[h_1(X)] = \mathbb{E}[X_1], \quad \mathbb{E}[h_2(X)] = \mathbb{E}[X_2] \end{aligned} \tag{13}$$

Observe that by linearity of expectation,

$$\mathbb{E}[X_i] = \mathbb{E}[\min(X_i, d_i)] + \mathbb{E}[(X_i - d_i)^+], \quad i = 1, 2$$

and

$$\mathbb{E}[h(X_i)] = \mathbb{E}[\min(X_i, d_i)] + \mathbb{E}[(X_j - d_j)^+], \quad i, j = 1, 2;\ i \neq j.$$

Therefore the fairness constraint implies

$$\mathbb{E}[h_i(X)] = \mathbb{E}[X_i] \iff \mathbb{E}[(X_1 - d_1)^+] = \mathbb{E}[(X_2 - d_2)^+]. \tag{14}$$

For the loss function, suppose that X_1, X_2 are independent. Then,

$$\operatorname{Var}[h_i(X)] = \operatorname{Var}[\min(X_i, d_i)] + \operatorname{Var}[(X_j - d_j)^+] + \underbrace{2\operatorname{Cov}[\min(X_i, d_i), (X_j - d_j)^+]}_{0}, \quad i, j = 1, 2; i \neq j.$$

Therefore, our cost function is equivalent to

$$\mathrm{Var}[h(X_1)] + \mathrm{Var}[h(X_2)] = \mathrm{Var}[\min(X_1, d_1)] + \mathrm{Var}[(X_2 - d_2)^+] + \mathrm{Var}[\min(X_2, d_2)] + \mathrm{Var}[(X_1 - d_1)^+].$$

Observe that

$$\mathrm{Var}[X_i] = \mathrm{Var}[\min(X_i, d_i) + (X_i - d_i)^+] = \mathrm{Var}[\min(X_i, d_i)] + \mathrm{Var}[(X_i - d_i)^+] + 2\,\mathrm{Cov}[\min(X_i, d_i), (X_i - d_i)^+]$$

$$\implies \mathrm{Var}[\min(X_i, d_i)] + \mathrm{Var}[(X_i - d_i)^+] = \mathrm{Var}[X_i] - 2\,\mathrm{Cov}[\min(X_i, d_i), (X_i - d_i)^+].$$

Plugging this into our cost function, we want to solve

$$\begin{aligned} \min_{d_1, d_2} \ & \mathrm{Var}[X_1] - 2\,\mathrm{Cov}[\min(X_1, d_1), (X_1 - d_1)^+] + \mathrm{Var}[X_2] - 2\,\mathrm{Cov}[\min(X_2, d_2), (X_2 - d_2)^+] \\ \text{s.t. } & \mathbb{E}[(X_1 - d_1)^+] = \mathbb{E}[(X_2 - d_2)^+]. \end{aligned} \tag{15}$$

The variance terms are constant with respect to d_1, d_2, the scaling of 2 does not impact our minimization, and the negation produces a maximization problem. Therefore, our problem simplifies to finding

$$\begin{aligned} \max_{d_1, d_2} \ & \mathrm{Cov}[\min(X_1, d_1), (X_1 - d_1)^+] + \mathrm{Cov}[\min(X_2, d_2), (X_2 - d_2)^+] \\ \text{s.t. } & \mathbb{E}[(X_1 - d_1)^+] = \mathbb{E}[(X_2 - d_2)^+]. \end{aligned} \tag{16}$$

Observe now that

$$\mathrm{Cov}[\min(X_i, d_i), (X_i - d_i)^+] = \mathbb{E}[\min(X_i, d_i)(X_i - d_i)^+] - \mathbb{E}[\min(X_i, d_i)]\mathbb{E}[(X_i - d_i)^+].$$

Note,

$$\min(X_i, d_i)(X_i - d_i)^+ = \begin{cases} 0 & X_i \le d_i \\ d_i(X_i - d_i) & X_i > d_i \end{cases} = d_i(X_i - d_i)^+$$

so

$$\mathrm{Cov}[\min(X_i, d_i), (X_i - d_i)^+] = (d_i - \mathbb{E}[\min(X_i, d_i)])\mathbb{E}[(X_i - d_i)^+].$$

Again we can plug this into the loss function and use the fairness constraint. Our final formulation of the optimization problem is to find

$$\begin{aligned} \max_{d_1, d_2} \ & \mathbb{E}[(X_1 - d_1)^+]\,(d_1 + d_2 - \mathbb{E}[\min(X_1, d_1)] - \mathbb{E}[\min(X_2, d_2)]) \\ \text{s.t. } & \mathbb{E}[(X_1 - d_1)^+] = \mathbb{E}[(X_2 - d_2)^+]. \end{aligned} \tag{17}$$

For arbitrary independent X_1, X_2, a solution does not exist so instead suppose that $X_1 \sim \mathrm{Exp}(\lambda_1)$, $X_2 \sim \mathrm{Exp}(\lambda_2)$ for some $\lambda_1, \lambda_2 > 0$. Then,

$$\begin{aligned} \mathbb{E}[(X - d)^+] &= \int_d^\infty (x - d)\lambda e^{-\lambda x} dx \\ &= \int_0^\infty u\lambda e^{-\lambda d} e^{-\lambda u} du \\ &= \Big[-u e^{-\lambda(u+d)}\Big]_0^\infty + \int_0^\infty e^{-\lambda(u+d)} du \\ &= \frac{e^{-\lambda d}}{\lambda}, \end{aligned}$$

$$\begin{aligned} \mathbb{E}[\min(X, d)] &= \int_0^d x\lambda e^{-\lambda x} dx + d\int_d^\infty \lambda e^{-\lambda x} dx \\ &= \left[-x e^{-\lambda x}\right]_0^d + \int_0^d e^{-\lambda x} dx + d\int_d^\infty \lambda e^{-\lambda x} \\ &= -d e^{-\lambda d} + \frac{1 - e^{-\lambda d}}{\lambda} + d e^{-\lambda d} \\ &= \frac{1 - e^{-\lambda d}}{\lambda}. \end{aligned}$$

From the constraint

$$\mathbb{E}[(X_1-d_1)^+]=\mathbb{E}[(X_2-d_2)^+] \implies \frac{e^{-\lambda_1 d_1}}{\lambda_1}=\frac{e^{-\lambda_2 d_2}}{\lambda_2}.$$

Taking the logarithm and solving for d_2, we have that

$$d_2=\frac{\lambda_1 d_1+\log(\lambda_1/\lambda_2)}{\lambda_2}. \tag{18}$$

Therefore, we can minimize directly over d_1 and use the above to compute d_2. Note that

$$\mathbb{E}[\min(X_2,d_2)]=\frac{1-e^{-\lambda_2 d_2}}{\lambda_2}=\frac{1-e^{-\lambda_1 d_1-\log(\lambda_1/\lambda_2)}}{\lambda_2}=\frac{1-e^{-\lambda_1 d_1}\frac{\lambda_2}{\lambda_1}}{\lambda_2}=\frac{1}{\lambda_2}-\frac{e^{-\lambda_1 d_1}}{\lambda_1}.$$

Putting everything together, the problem to minimize can be written

$$\begin{aligned}
&\frac{e^{-\lambda_1 d_1}}{\lambda_1}\left(d_1+\frac{\lambda_1 d_1+\log(\lambda_1/\lambda_2)}{\lambda_2}-\frac{1-e^{-\lambda_1 d_1}}{\lambda_1}-\frac{1}{\lambda_2}+\frac{e^{-\lambda_1 d_1}}{\lambda_1}\right)\\
&\quad=\frac{e^{-\lambda_1 d_1}}{\lambda_1}\left(d_1+\frac{\lambda_1}{\lambda_2}d_1+\frac{\log(\lambda_1/\lambda_2)}{\lambda_2}-\frac{1}{\lambda_1}-\frac{1}{\lambda_2}+\frac{2e^{-\lambda_1 d_1}}{\lambda_1}\right)\\
&\quad=e^{-\lambda_1 d_1}\left(\underbrace{\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)}_{C_1}d_1+\underbrace{\frac{\log(\lambda_1/\lambda_2)}{\lambda_1\lambda_2}-\frac{1}{\lambda_1^2}-\frac{1}{\lambda_1\lambda_2}}_{C_2}+\underbrace{\frac{2}{\lambda_1^2}}_{C_3}e^{-\lambda_1 d_1}\right)\\
&\quad\equiv C_1 d_1 e^{-\lambda_1 d_1}+C_2 e^{-\lambda_1 d_1}+C_3 e^{-2\lambda_1 d_1}.
\end{aligned}$$

Therefore, we wish to solve

$$\begin{aligned}
&\min_{d_1}\, C_1 d_1 e^{-\lambda_1 d_1}+C_2 e^{-\lambda_1 d_1}+C_3 e^{-2\lambda_1 d_1},\\
&\text{where }\; C_1=\frac{1}{\lambda_1}+\frac{1}{\lambda_2},\quad C_2=\frac{\log(\lambda_1/\lambda_2)}{\lambda_1\lambda_2}-\frac{1}{\lambda_1^2}-\frac{1}{\lambda_1\lambda_2},\quad C_3=\frac{2}{\lambda_1^2}.
\end{aligned} \tag{19}$$

This is a smooth function with at least one point in which the derivative vanishes and is also a local maximum. Moreover, it is unimodal so this local maximum is a global maximum. Differentiating, we have

$$0=C_1(1-d_1\lambda_1)e^{-\lambda_1 d_1}-C_2\lambda_1 e^{-\lambda_1 d_1}-2C_3\lambda_1 e^{-2\lambda_1 d_1}$$

which implies

$$e^{\lambda_1 d_1}[C_1(1-\lambda_1 d_1)-C_2\lambda_1]=2C_3\lambda_1. \tag{20}$$

It is not possible to obtain an explicit formula for d_1 from here so we must use a root finding algorithm to solve for d_1. We summarize the results in the following theorem:

Theorem 4.1.1. *Let $X_1\sim\mathrm{Exp}(\lambda_1)$, $X_2\sim\mathrm{Exp}(\lambda_2)$ be independent and define the general rule*

$$h_i(X)=\min(X_i,d_i)+(X_j-d_j)^+,\quad i,j=1,2;\; i\neq j \tag{21}$$

for some deductibles $d_1,d_2\in\mathbb{R}$. Then, d_1^*,d_2^* solve the optimization problem

$$\begin{aligned}
\min_{d_1,d_2}\;&\mathrm{Var}(h_1(X))+\mathrm{Var}(h_2(X))\\
\textit{s.t.}\;&\mathbb{E}[h_i(X)]=\mathbb{E}[X_i],\quad i=1,2
\end{aligned} \tag{22}$$

if and only if they satisfy

$$e^{\lambda_1 d_1^*}[C_1(1-\lambda_1 d_1^*)-C_2\lambda_1]=2C_3\lambda_1,\quad d_2^*=\frac{\lambda_1 d_1^*+\log(\lambda_1/\lambda_2)}{\lambda_2} \tag{23}$$

where

$$C_1=\frac{1}{\lambda_1}+\frac{1}{\lambda_2},\quad C_2=\frac{\log(\lambda_1/\lambda_2)}{\lambda_1\lambda_2}-\frac{1}{\lambda_1^2}-\frac{1}{\lambda_1\lambda_2},\quad C_3=\frac{2}{\lambda_1^2}. \tag{24}$$

Remark 4.1.2. In the case $\lambda = \lambda_1 = \lambda_2$, that is $X_1, X_2 \sim \text{Exp}(\lambda)$, we have

$$d_2^* = \frac{\lambda d_1^* + \log(1)}{\lambda} = d_1^*$$

and

$$C_1 = \frac{2}{\lambda}, \quad C_2 = -\frac{2}{\lambda^2}, \quad C_3 = \frac{2}{\lambda^2}.$$

Then, the condition on d_1^* reduces to

$$e^{\lambda d_1^*}\left[\frac{2}{\lambda}(1-\lambda d_1^*) + \frac{2}{\lambda}\right] = \frac{4}{\lambda} \implies e^{\lambda d_1^*}\left[1 - \frac{\lambda d_1^*}{2}\right] = 1.$$

Let $y = \lambda d_1^* - 2$. Then,

$$e^{y+2}\left[1 - \frac{y+2}{2}\right] = 1 \implies ye^y = -2e^{-2}$$

so

$$y = W(-2e^{-2})$$

where W is the Lambert-W function. Therefore,

$$d_1^* = d_2^* = \frac{W(-2e^{-2}) + 2}{\lambda}.$$

4.1.2 n Agents

We begin with an alternative perspective of the case with two agents. Recall:

$$X_i = \min(X_i, d_i) + (X_i - d_i)^+, \quad i = 1, 2$$

$$h_i(X) = \min(X_i, d_i) + (X_j - d_j)^+, \quad i, j = 1, 2;\ i \neq j$$

Let $X = (X_1, X_2)^T$, $d = (d_1, d_2)^T$ and define $\min(X, d), (X - d)^+$ to be pointwise operations. Then,

$$h(X) = \min(X, d) + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}(X - d)^+ = X - (X - d)^+ + \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P}(X - d)^+.$$

Definition 4.1.3. *Let $P \in \mathbb{R}^{n \times n}$. Then P is said to be a **signed column-stochastic** if and only if*

$$\sum_{i=1}^{n} P_{ij} = 1 \iff \mathbf{1}^T P = \mathbf{1}^T. \tag{25}$$

Note that here we use the term *signed* to refer to the fact that the entries can be negative. Observe that in the two agent case, P is indeed (signed) column-stochastic. We will see that this is a necessary property later for an arbitrarily large number of agents.

Definition 4.1.4. *We say that $P \in \mathbb{R}^{n \times n}$ is a **redistribution matrix** if and only if P is signed column-stochastic and $P_{ii} = 0$ for all $i = 1, \ldots, n$. For a given redistribution matrix $P \in \mathbb{R}^{n \times n}$ and deductible $d \in \mathbb{R}^N$, we define the **multi-agent risk sharing rule***

$$h(X) = X + (P - I)(X - d)^+. \tag{26}$$

Lemma 4.1.5. *The multi-agent risk sharing rule is conservative, that is, $\mathbf{1}^T h(X) = \mathbf{1}^T X$.*

Proof. We have $\mathbf{1}^T h(X) = \mathbf{1}^T X + \mathbf{1}^T (P - I)(X - d)^+$, so we verify $\mathbf{1}^T (P - I)(X - d)^+ = 0$ for any redistribution matrix P and deductible d. In fact, we can compute

$$\mathbf{1}^T (P - I)(X - d)^+ = \mathbf{1}^T P (X - d)^+ - \mathbf{1}^T (X - d)^+ = \mathbf{1}^T (X - d)^+ - \mathbf{1}^T (X - d) = 0$$

where we used that $\mathbf{1}^T P = \mathbf{1}^T$ since P is signed column-stochastic. □

Let us count the number of parameters we need. The matrix $P \in \mathbb{R}^{n\times n}$ has n^2 entries with zeros on the diagonal which leaves $n^2 - n$ potentially non-zero entries. For each column, we need the $n-1$ non-diagonal entries to sum to 1 which implies that each column has $n-1-1 = n-2$ degrees of freedom. Thus, P has $n(n-2)$ degrees of freedom. Since $d \in \mathbb{R}^N$ we have n degrees of freedom in d and so $n(n-2)+n = n(n-1)$ total degrees of freedom in this problem. For the two agent case ($n=2$), we see that P is completely determined and only $d = (d_1, d_2)^T$ must be solved for. More generally, however, we can develop an optimization problem of both variables.

Remark 4.1.6. We impose our expectation condition but adapt it to this matrix problem:

$$\mathbb{E}[h(X)] = \mathbb{E}[X] + (P-I)\mathbb{E}[(X-d)^+]$$

which implies

$$\mathbb{E}[h(x)] = \mathbb{E}[X] \iff (P-I)\mathbb{E}[(X-d)^+] = 0. \tag{27}$$

The matrix equation $(P-I)\mathbb{E}[(X-d)^+] = 0$ admits non-unique solutions when one exists. Indeed, since $1^T P = 1^T$ by column-stochasticity, we have $1^T(P-I) = 1^T P - 1^T I = 1^T - 1^T = 0$ and so $1 \in \mathrm{Ker}((P-I)^T)$. This implies $P-I$ has non-trivial kernel and therefore this is a valid constraint to impose.

We are now able to consider our variance minimization problem. The covariance-variance matrix of h is the symmetric matrix

$$\begin{aligned}\mathrm{Cov}(h(X)) = \mathrm{Cov}(X) &+ (P-I)\,\mathrm{Cov}((X-d)^+)(P-I)^T \\ &+ \mathrm{Cov}(X,(X-d)^+)(P-I)^T + (P-I)\,\mathrm{Cov}((X-d)^+, X).\end{aligned}$$

Here, we use the notation that for a random vector $Y = (Y_1, \ldots, Y_n)^T$, $\mathrm{Cov}(Y)_{ij} = \mathrm{Cov}(Y_i, Y_j)$. In particular, for $A = P - I$ and $Z = (X-d)^+$, the trace is given by

$$\mathrm{tr}(\mathrm{Cov}(h(X))) = \mathrm{tr}(\mathrm{Cov}(X)) + \mathrm{tr}(A\,\mathrm{Cov}(Z)A^T) + 2\,\mathrm{tr}(A\,\mathrm{Cov}(X,Z))$$

Note that $\sum_{i=1}^n \mathrm{Var}(h_i(X)) = \mathrm{tr}(\mathrm{Cov}(h(X)))$ and that $\mathrm{tr}(\mathrm{Cov}(X))$ does not depend on P or d so minimizing the variance of h like before is equivalent to minimizing

$$\mathrm{tr}(B\,\mathrm{Cov}(Z)B^T) + 2\,\mathrm{tr}(B\,\mathrm{Cov}(X,Z)).$$

Therefore, our multi-agent optimization problem is to solve

$$\begin{aligned}\min_{P\in\mathbb{R}^{n\times n} d\in\mathbb{R}^n} \quad & \mathrm{tr}((P-I)\,\mathrm{Cov}((X-d)^+)(P-I)^T) + 2\,\mathrm{tr}\left((P-I)\,\mathrm{Cov}(X,(X-d)^+)\right) \\ \text{s.t.} \quad & \mathbf{1}^T P = \mathbf{1}^T, \\ & (P-I)\mathbb{E}[(X-d)^+] = 0, \\ & \mathrm{diag}(P) = 0.\end{aligned} \tag{28}$$

For compactness, we define $A = P - I$, $\Sigma_d = \mathrm{Cov}((X-d)^+)$, $\Lambda_d = \mathrm{Cov}(X,(X-d)^+)$, and $\mu_d = \mathbb{E}[(X-d)^+]$. Adjusting the constraints, we now wish to solve

$$\begin{aligned}\min_{A\in\mathbb{R}^{n\times n},\ d\in\mathbb{R}^n} \quad & \mathrm{tr}(A\Sigma_d A^T) + 2\,\mathrm{tr}(A\Lambda_d) \\ \text{s.t.} \quad & \mathbf{1}^T A = 0, \\ & A\mu_d = 0, \\ & \mathrm{diag}(A) = -\mathbf{1}.\end{aligned} \tag{29}$$

For well-posedness, we need the following result.

Lemma 4.1.7. *For any real valued random variable X and deductible $d \in \mathbb{R}$,*

$$\mathrm{Var}((X-d)^+) = 0 \iff \mathbb{P}[X > d] = 0. \tag{30}$$

Proof. Let $Z = (X-d)^+$ and note that

$$\mathrm{Var}(Z) = \mathbb{E}[Z^2] - (\mathbb{E}[Z])^2$$

so

$$Z = 0 \iff X \le d, \quad Z > 0 \iff X > d.$$

Assume that $\mathbb{P}[X > d] = 0$. Then, $X \leq d$ almost surely which implies $Z = 0$ almost surely. Therefore, $Z = 0$ with probability 1 and Z is a constant random variable, that is, $\mathrm{Var}(Z) = 0$. Assume now that $\mathrm{Var}(Z) = 0$. Then Z is constant almost surely and $Z = (X - d)^+ \geq 0$ always. The only valid constant is 0. Indeed, suppose that $Z = c > 0$. Then, $X > d$ almost surely which is not true in general. So

$$Z = 0 \text{ a.s.} \implies \mathbb{P}[Z > 0] = 0 \implies \mathbb{P}[X > d] = 0.$$

as desired. □

Assume that $X_1, \ldots, X_n$ are independent and chosen from unbounded (possibly identical) distributions. Then,

$$\Sigma_d = \mathrm{diag}(\mathrm{Var}((X_1 - d_1)^+), \ldots, \mathrm{Var}((X_n - d_n)^+)) \tag{31}$$

and so Σ_d is positive definite if and only if $\mathrm{Var}((X_i - d_i)^+) > 0$ for all $i = 1, \ldots, n$. By a contrapositive of the previous lemma, we have

$$\mathrm{Var}((X - d)^+) > 0 \iff \mathbb{P}[X > d] > 0. \tag{32}$$

and so it suffices to show that $\mathbb{P}[X > d] > 0$. Indeed, when each X_i has unbounded support from above, then $\mathbb{P}[X_i > d_i] > 0$ for all $d_i \in \mathbb{R}$ and for all $i = 1, \ldots, n$. If X_i has bounded support, then Σ_d is positive definite for $d < \sup \mathrm{supp}(X)$. While we formulated the initial problem in $\mathbb{R}^n$, we must consider a sufficiently sized compact domain instead to obtain d^*. This does not present an issue in practice as d^* will be a finite value. We shall see later that this assumption is necessary as well. Therefore, we can assume Σ_d to be positive definite. We now fix $d \in [a, b]^n$ and consider

$$f(A; d) = \mathrm{tr}(A\Sigma_d A^T) + 2\,\mathrm{tr}(A\Lambda_d) \tag{33}$$

This function is strictly convex in A because $\mathrm{tr}(A\Sigma_d A^T)$ is a strictly convex quadratic form and $\mathrm{tr}(A\Lambda_d)$ is linear. Furthermore, the constraint set

$$\mathbf{1}^T A = 0, \quad A\mu_d = 0, \quad \mathrm{diag}(A) = -\mathbf{1}$$

is affine. Therefore, for each fixed $d \in [a, b]^n$, the constrained minimization over A has a unique solution $A^*(d)$.

We will now construct such a matrix using Lagrange multipliers. Let $d \in [a, b]^n$ be fixed with associated $\Sigma_d \in \mathbb{R}^{n\times n}$ diagonal and positive definite, $\Gamma_d \in \mathbb{R}^{n\times n}$, and $\mu_d \in \mathbb{R}^n$. We wish to solve

$$\begin{aligned} \min_{A\in\mathbb{R}^{n\times n},\ d\in[a,b]^n} \quad & \mathrm{tr}(A\Sigma_d A^T) + 2\,\mathrm{tr}(A\Lambda_d) \\ \text{s.t.} \quad & \mathbf{1}^T A = 0, \\ & A\mu_d = 0, \\ & \mathrm{diag}(A) = -\mathbf{1} \end{aligned} \tag{34}$$

For the last set of constraints on the diagonal entries, we set up the Lagrange multipliers $\{\gamma_i\}$ so that the term $\sum_{i=1}^n \gamma_i(A_{ii} + 1)$ appears in the Lagrangian. We introduce Γ so that for $\lambda, \beta, \gamma \in \mathbb{R}^n$ and $\mathrm{diag}(\Gamma) = \gamma$, the associated Lagrangian of this problem is

$$\mathcal{L}(A; \lambda, \beta, \Gamma) = \mathrm{tr}(A\Sigma_d A^T) + 2\,\mathrm{tr}(A\Lambda_d) + (\mathbf{1}^T A)\lambda + \beta^T A\mu_d + \mathrm{tr}(\Gamma A) + \mathrm{tr}(\Gamma). \tag{35}$$

Note that

$$\mathbf{1}^T A\lambda = \lambda^T A^T \mathbf{1} = \mathrm{tr}(\lambda^T A^T \mathbf{1}) = \mathrm{tr}(A^T \mathbf{1}\lambda^T)$$

and

$$\beta^T A\mu_d = \mathrm{tr}(A\mu_d\beta^T)$$

where we used that for any $u, v \in \mathbb{R}^n$, $\mathrm{tr}(uv^T) = \mathrm{tr}(v^T u) = v^T u$. Therefore, our Lagrangian is equivalent to

$$\mathcal{L}(A; \lambda, \beta, \Gamma) = \mathrm{tr}(A\Sigma_d A^T) + 2\,\mathrm{tr}(A\Lambda_d) + \mathrm{tr}(A^T\mathbf{1}\lambda^T) + \mathrm{tr}(A\mu_d\beta^T) + \mathrm{tr}(\Gamma A) + \mathrm{tr}(\Gamma). \tag{36}$$

Differentiating term by term with respect to A, we compute that

$$\begin{aligned} \nabla_A\, \mathrm{tr}(A\Sigma_d A^T) &= 2A\Sigma_d, \\ \nabla_A 2\,\mathrm{tr}(A\Lambda_d) &= 2\Lambda_d^T, \\ \nabla_A\, \mathrm{tr}(A^T\mathbf{1}\lambda^T) &= \mathbf{1}\lambda^T, \\ \nabla_A\, \mathrm{tr}(A\mu_d\beta^T) &= \beta\mu_d^T, \\ \nabla_A\, \mathrm{tr}(\Gamma A) &= \Gamma. \end{aligned}$$

and so the first variation and necessary condition $\nabla_A \mathcal{L}(A^*(d)) = 0$ then implies that

$$2A^*(d)\Sigma_d + 2\Lambda_d^T + \mathbf{1}\lambda^T + \beta\mu_d^T + \Gamma = 0.$$

Since Σ_d is positive definite,

$$A^*(d) = -\Lambda_d^T \Sigma_d^{-1} - \frac{1}{2}\mathbf{1}\lambda^T \Sigma_d^{-1} - \frac{1}{2}\beta\mu_d^T \Sigma_d^{-1} - \frac{1}{2}\Gamma\Sigma_d^{-1}. \tag{37}$$

Plugging $A^*(d)$ into the constraints, we obtain the three systems for λ, β, Γ

$$-\mathbf{1}^T \Lambda_d^T \Sigma_d^{-1} - \frac{1}{2}\mathbf{1}^T \mathbf{1}\lambda^T \Sigma_d^{-1} - \frac{1}{2}\mathbf{1}^T \beta\mu_d^T \Sigma_d^{-1} - \frac{1}{2}\mathbf{1}^T \Gamma\Sigma_d^{-1} = 0, \tag{38}$$

$$-\Lambda_d^T \Sigma_d^{-1}\mu_d - \frac{1}{2}\mathbf{1}\lambda^T \Sigma_d^{-1}\mu_d - \frac{1}{2}\beta\mu_d^T \Sigma_d^{-1}\mu_d - \frac{1}{2}\Gamma\Sigma_d^{-1}\mu_d = 0, \tag{39}$$

$$\operatorname{diag}\left(-\Lambda_d^T \Sigma_d^{-1} - \frac{1}{2}\mathbf{1}\lambda^T \Sigma_d^{-1} - \frac{1}{2}\beta\mu_d^T \Sigma_d^{-1} - \frac{1}{2}\Gamma\Sigma_d^{-1}\right) = -\mathbf{1}. \tag{40}$$

This $3n \times 3n$ system can be solved for $(\lambda^*, \beta^*, \Gamma^*) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times n}$ where $\gamma^* = \Gamma_{ii}^*$.

With an explicit construction of the minimizer $A^*(d)$ for arbitrary $d \in [a,b]^n$, we can now minimize on d. Define

$$\phi(d) = f(A^*(d), d) = \operatorname{tr}(A^*(d)\operatorname{Cov}((X-d)^+)A^*(d)^T) + 2\operatorname{tr}(A^*(d)\operatorname{Cov}(X,(X-d)^+)). \tag{41}$$

The function $\phi(d)$ is continuous on $\mathbb{R}^n$ so for sufficiently large compact regions $[a,b]^n$,

$$\min_{d\in[a,b]^n} \phi(d) \tag{42}$$

has a global minimizer by the extreme value theorem. Because this problem is nonlinear and in general non-convex, a numerical solver must be used to physically compute d^*. To summarize, we present the following theorem.

Theorem 4.1.8. *There exists a $d^* \in [a,b]^n$ and corresponding $A^* \in \mathbb{R}^{n\times n}$ that solve*

$$\begin{aligned} \min_{A\in\mathbb{R}^{n\times n},\, d\in[a,b]^n} \quad & \operatorname{tr}(\operatorname{Cov}(h(X))) \\ \textit{s.t.} \quad & \mathbf{1}^T A = 0, \\ & A\mu_d = 0, \\ & \operatorname{diag}(A) = -\mathbf{1} \end{aligned} \tag{43}$$

We may also recover $P^* = A^* + I$ for determining the optimal allocation matrix.

Remark 4.1.9. Uniqueness of the global solution depends on the strong convexity and coercivity of $f(A(d), d)$. In general, this is not attained but may hold for sufficiently nice distributions.

Remark 4.1.10. We can impose non-negativity of entries in P, which in turn implies $0 \le P_{ij} \le 1$ for all $i, j = 1, \ldots, n$ because P is column-stochastic. This means that agent i can cover up to 100% of agent j's loss but never more and never a negative amount. This prevents individuals from profiting. or charging someone's excess back to them. For brevity, we use the notation $0 \le P \le 1$. In this case, the problem is generally not solvable.

To test our approach, we consider $X_1 \sim \operatorname{Exp}(1)$, $X_2 \sim \operatorname{Exp}(0.8)$, $X_3 \sim \operatorname{Exp}(1.2)$ for 100000 samples. We let $d = (1, 1, d_3)$ and vary $d_3 \in (0, 5]$. We compute $A^*(d)$ (equivalently $P^*(d)$ through a direct minimization of

$$\begin{aligned} \min_{A(d)\in\mathbb{R}^{n\times n}} \quad & \operatorname{tr}(\operatorname{Cov}(h(X))) \\ \text{s.t.} \quad & \mathbf{1}^T A = 0, \\ & A\mu_d = 0, \\ & \operatorname{diag}(A) = -\mathbf{1}, \end{aligned}$$

and through our proposed approach based on Lagrange multipliers. Both methods are compared based on their computed value of $\operatorname{tr}(\operatorname{Cov}(h(X)))$. We also examine the ability of both methods to enforce the constraints by examining $\|P\mu_d - \mu_d\|_2$, $\|\operatorname{diag}(P)\|_2$, and $\|\mathbf{1}^T P - \mathbf{1}^T\|_2$.

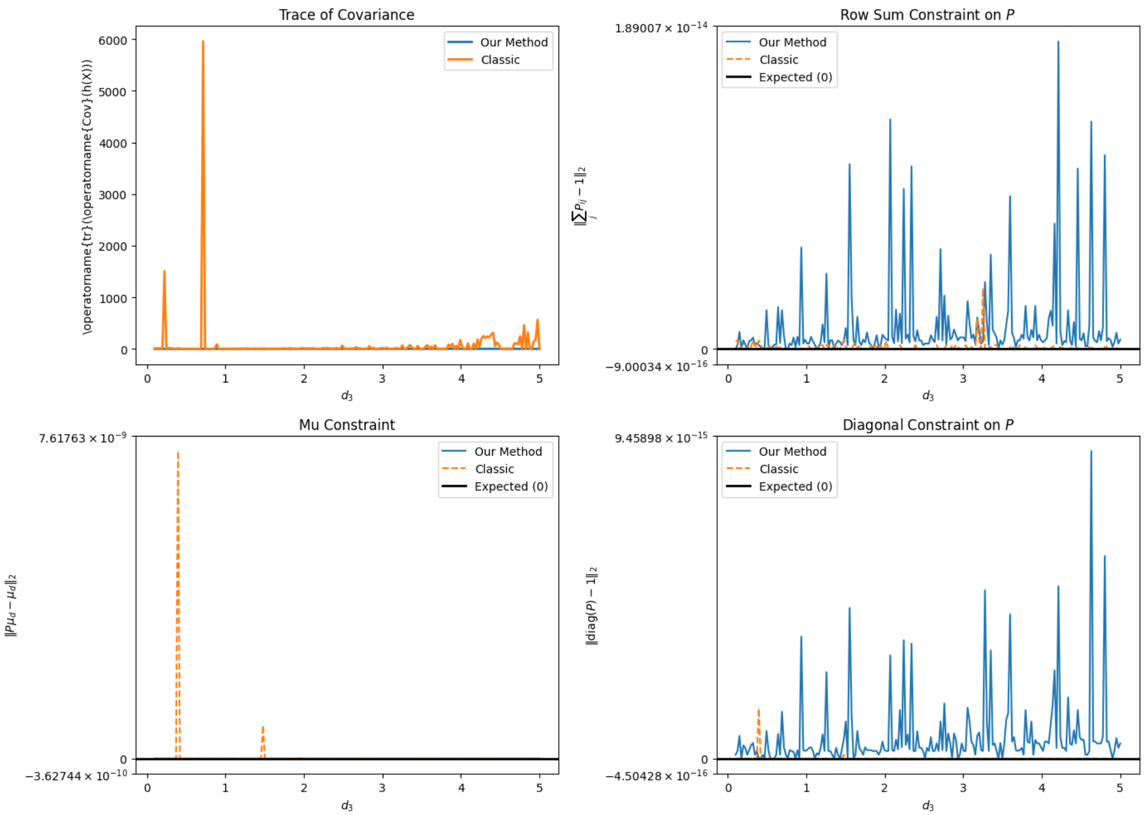


Figure 4: A comparison of the classical optimization and our proposed method. Both methods enforce the constraints on the level of machine epsilon. However, our proposed approach successful computes the minimum value in comparison to the classic solver which frequently diverges.

4.2 Exponential deductible

For simplicity, the concept of sharing rules would be presented here for the case of agents. The distribution rules are as follows:

$$\begin{cases} H_1(X_1, X_2) = h_1(X_1) + X_2 - h_2(X_2), \\ H_2(X_1, X_2) = h_2(X_2) + X_1 - h_1(X_1). \end{cases}$$

In this subsection, we explore other new distribution rules for h_1 and h_2. Most of the rules could be applied for equivalent functions h_1 and h_2, that is, $h_1(x) = h_2(x)$ for all $x \geq 0$.

Proposition 4.2.1. *If a system with two agents has equivalent local gain rules $h_1(X) = h_2(X)$ and $E[h_i(X)] = E[X]$ for all i, then the fair distribution rule is satisfied, i.e. $E[H_1(X_1, X_2)] = E[X_1]$ and $E[H_2(X_1, X_2)] = E[X_2]$.*

Proof. Here,

$$E[H_1(X_1, X_2)] = E[h_1(X_1) + X_2 - h_2(X_2)] = E[h_1(X_1)] + E[X_2] - E[h_2(X_2)] =$$
$$= E[X_1] + E[X_2] - E[X_2] = E[X_1].$$

Similarly it can be shown that $E[H_2(X_1, X_2)] = E[X_2]$. □

In the case of ordinary deductible from the previous subsection, there is an issue of differentiability at the point d. The lack of smoothness may lead to the peers experiencing sharp changes in the region around d. In this subsection, the problem is further studied and a possible approach is presented. For small amounts of loss, say in the interval $[0, d]$, we use the same mapping as in the ordinary deductible case. For amounts of loss exceeding d, a more smooth solution is provided. Let the remaining part of the mapping be $f(x)$ for $x \in [d, \infty)$. Then $f(x)$ should satisfy the following smoothness conditions:

1. $f'(d) = 1$. This ensures that the derivative of f at d is smooth.
2. $f(d) = d$. This constraint arises directly from the derivative condition above.
3. $f(x)$ should be from the class $f(x) = k - \beta e^{-\alpha x}$. A different class of functions could also be chosen as long as it results in smoothness.
4. $\lim_{x\to\infty} f(x) = \omega$, where ω is the maximum amount the agent would pay for its own loss.

The four conditions above give rise to

$$f(x) = \omega - d - (\omega - 2d)e^{\frac{d}{\omega-2d}} e^{-\frac{1}{\omega-2d}x}, \quad x \geq d$$

The mapping $h(x)$ is:

$$h(X) = \begin{cases} x, & for x \leq d \\ \omega - d - (\omega - 2d)e^{\frac{d-x}{\omega-2d}}, & for x > d. \end{cases}$$

An illustration is provided in Figure 5.

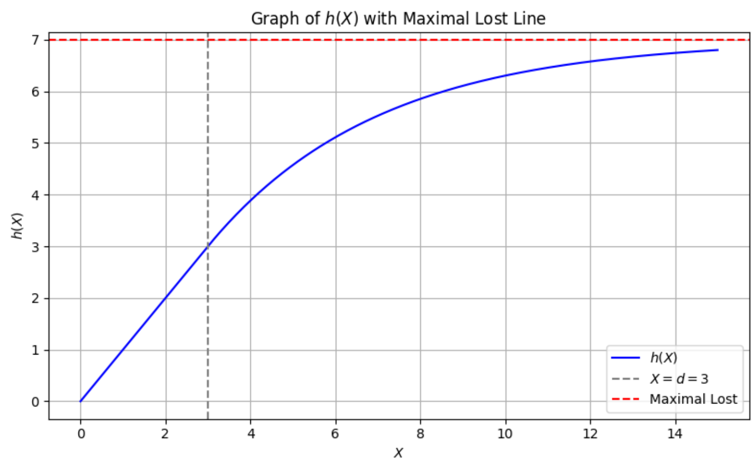


Figure 5: Exponential loss function

Now, we caculate some expected values. Let $f(x)$ be the probability density function of X. Then,

$$\begin{aligned} E[h(X)] &= \int_0^d x \cdot f(x)\, dx + \int_d^\infty (\omega - d - (\omega - 2d)e^{\frac{d-x}{\omega-2d}}) f(x)\, dx \\ &= E[X] + \int_d^\infty (\omega - d - (\omega - 2d)e^{\frac{d-x}{\omega-2d}} - x) f(x)\, dx \end{aligned}$$

Assume that X is exponentially distributed with mean $\frac{1}{\lambda}$, that is,

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0, \\ 0, & x < 0. \end{cases}$$

It can be shown that the integral in $E(h(X))$ becomes

$$\begin{aligned} I(d, \omega, \lambda) &\equiv \int_d^\infty (\omega - d - (\omega - 2d)e^{\frac{d-x}{\omega-2d}} - x) f(x)\, dx \\ &= (\omega - d)(e^{-\lambda d}) + (\omega - 2d)e^{\frac{d}{\omega-2d}} \lambda \left(-\frac{\omega - 2d}{1 + \lambda\omega - 2\lambda d} \right) (e^{-\frac{1+\lambda\omega-2\lambda d}{\omega-2d}d}) - e^{-\lambda d} \end{aligned}$$

The equation for the relation of the expected values can now be represented by

$$E[h(X)] = E[X] + I(d, \omega, \lambda).$$

To satisfy the fairness rule $E[h(X)] = E[X]$, we require

$$I(d, \omega, \lambda) = 0$$

In this risk sharing scheme, we can control d and ω, but not λ. Therefore, we can write

$$d = G(\omega),$$

where $G(\omega)$ is the implicitly defined by the equation $I(G(\omega), \omega, \lambda) = 0$, which can be determined numerically.

4.3 Quantile risk-sharing

4.3.1 α quantile and comonotonicity

We define the cumulative distribution function (CDF) as $F(x) = \mathbb{P}(X \leq x)$ and the inverse of the CDF gives us the quantile function

$$F^{-1}(p) = \min\{x \in \mathbb{R} | F_X(x) \geq p\}, p \in (0, 1).$$

Sometimes, $F^{-1}(p)$ is also called a quantile of order p.

For a more precise definition that resolves the ambiguity at discontinuities in quantiles, Dhaene et al. [2025] suggested α-quantile function of order p to be:

$$F^{-1}_{X(\alpha)}(p) = \begin{cases} F_X^{-1+}(0) & \text{if } p = 0, \\ \alpha F_X^{-1}(p) + (1-\alpha) F_X^{-1+}(p) & \text{if } p \in (0, 1), \\ F_X^{-1}(1) & \text{if } p = 1. \end{cases}$$

where $F_X^{-1}(p) = \inf\{x \in \mathbb{R} | F_X(x) \geq p\}$, $F_X^{-1+}(p) = \sup\{x \in \mathbb{R} | F_X(x) \leq p\}$. The term α is a parameter that determines the quantile of an agent at discontinuity.

Comonotonicity is another important concept that helps define quantile risk-sharing: A random vector $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is comonotonic if there exist non-decreasing functions $g_i : \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathbf{X} = (g_1(S_X), g_2(S_x), \ldots, g_n(S_x))$. In terms of quantile functions of order *U*, $\mathbf{X}^c$ is comonotonic (or the comonotonic counterpart of **X**) if $\mathbf{X}^c = \left(F^{-1}_{X_1}(U), F^{-1}_{X_2}(U), \ldots, F^{-1}_{X_n}(U)\right)$.

For example, consider the vector $\vec{X} = (X_1, X_2, X_3)$, with densities $f_1 = e^x$, $f_2 = \frac{1}{2}e^{-\frac{x}{2}}$, and $f_3 = \frac{1}{3}e^{-\frac{x}{3}}$, respectively. Then, the comonotonic vector $\vec{X_3^C}$ contains the quantile functions for X_1, X_2, and X_3. That is, $\vec{X_3^C} = (F_1^{-1}, F_2^{-1}, F_3^{-1}) = \left(\ln x, \frac{-2\ln x}{\ln(\frac{1}{2})+1}, \frac{-3\ln x}{\ln(\frac{1}{3})+1}\right)$. Furthermore, the comonotonic sum $S_3^C = F_1^{-1} + F_2^{-1} + F_3^{-1} = \ln x + \frac{-2\ln x}{\ln(\frac{1}{2})+1} + \frac{-3\ln x}{\ln(\frac{1}{3})+1}$.

Comonotonicity ensures that all agents in the sharing pool move together; if one agent faces a high loss, so do all the others. This does not necessarily mean that the total risk is being shared fairly.

4.3.2 Quantile risk-sharing rule

Quantile risk-sharing rule is defined as a risk-sharing rule h for any pool X, where the contribution for individual i is given by $h_i(X) = h_i(S_n) = F_i^{-1(\alpha_{S_n})}(F_{S_n^c}(S_n)), i = 1, 2, \ldots, n$.

The idea of joining the pool based on quantile risk-sharing is to replace the possibility of different p_i by unique and uniform probability levels. The solution is to ask participant i to contribute the amount $F_X^{-1}(p(s))$ where $p(s)$ satisfies $\sum_{i=1}^n F^{-1}_{X_i}(p_i) = \sum_{i=1}^n F^{-1}_{X_i}(p(s)) = s$. In other words, given the individual's risk level and the total loss realized, we can calculate the quantile level for all agents. As a result, each agent will be assigned a dollar amount based on their own risk distribution $F^{-1}_{X_i}(p(S_n))$.

The quantile risk-sharing rule satisfies the reshuffling, normalization, translativity, positive homogeneity, constancy, no-ripoff, comonotic, stand-alone for comotonic losses, and uniform for exchangeable losses properties (Denuit et al. [2022]). Additionally, we have that a risk-sharing rule $\mathbb{H}$ is the quantile risk-sharing rule if and only if $\mathbb{H}$ is aggregate, dependence-free, and stand-alone for comonotonic pools (Dhaene et al. [2025]).

4.3.3 Actuarial fairness

The quantile risk-sharing rule does not necessarily satisfy the actuarial fairness (Denuit et al. [2022]). For example, let $n = 2$, X_1 have distribution function F, and X_2 have distribution function $F \circ g^{-1}$, where g is a continuous, positive and increasing function. Consider the vector of losses $\vec{X} = (X_1, X_2)$. Then,

$$\begin{aligned} P[H_1^{\text{quant}}(\vec{X_2}) > v] &= P[S_2 > (F^{-1} + g \circ F^{-1}) \circ F(v)] \\ &= P[S_2 > F^{-1} \circ F(v) + (g \circ F^{-1}) \circ F(v)] \\ &= P[S_2 > v + g(v)] \\ &= P[h^{-1}(S_2) > v], \end{aligned}$$

where $h(v) = v + g(v)$. Therefore,

$$\begin{aligned} \mathbb{E}[H_1^{quant}(\vec{X_2})] &= \mathbb{E}[h^{-1}(S_2)] \\ &= \mathbb{E}[h^{-1}(X_1 + g(Z_1))] \\ &= \mathbb{E}[h^{-1}(X_1 - Z_1 + h(Z_1))]. \end{aligned}$$

The second equality is true because $S_2 = X_1 + X_2 = X_1 + g(Z_1)$.

We want $\mathbb{E}[H_1^{quant}(\vec{X_2})] = \mathbb{E}[X_1]$ so that can say that the quantile risk-sharing rule is fair. In other words, we want $\mathbb{E}[h^{-1}(X_1 - Z_1 + h(Z_1))] = \mathbb{E}[X_1]$. This will only happen if g is linear or if X_1 and Z_1 are comonotonic. Let us look at the case where g is linear. Then, $g(Z_1) = aZ_1$, where $a \in \mathbb{R}$. Hence, $h(v) = v + g(v) = v + av = (1 + a)v$ and $h^{-1}(v) = \dfrac{v}{1+a}$. Thus,

$$\begin{aligned} \mathbb{E}[h^{-1}(X_1 - Z_1 + h(Z_1))] &= \mathbb{E}[h^{-1}(X_1 - Z_1 + (1 + a)(Z_1)] \\ &= \mathbb{E}[h^{-1}(X_1 + aZ_1)] \\ &= \mathbb{E}\left[\frac{X_1 + aZ_1}{1+a}\right] \\ &= \frac{1}{1+a}\mathbb{E}[X_1] + \frac{a}{1+a}\mathbb{E}[Z_1] \\ &= \frac{1}{1+a}\mathbb{E}[X_1] + \frac{a}{1+a}\mathbb{E}[X_1] \\ &= \mathbb{E}[X_1]. \end{aligned}$$

Notice that $\mathbb{E}[X_1] = \mathbb{E}[Z_1]$ because X_1 and Z_1 are comonotonic.

Now consider the second case where X_1 and Z_1 are comonotonic. Then,

$$\begin{aligned} \mathbb{E}[h^{-1}(X_1 - Z_1 + h(Z_1))] &= \mathbb{E}[h^{-1}(X_1 - X_1 + h(X_1))] \\ &= \mathbb{E}[h^{-1}(h(X_1))] \\ &= \mathbb{E}[X_1]. \end{aligned}$$

However, if X_1 and Z_1 are not comonotonic and g is nonlinear, then $\mathbb{E}[H_1^{quant}(\vec{X_2})] \neq \mathbb{E}[X_1]$. Thus, the quantile risk-sharing rule would not satisfy the actuarial fairness property.

4.3.4 Examples

To better understand the definitions and the scheme, the examples below showcase two types of transformation which differ in terms of actuarial fairness. For both examples, let us consider $n = 2$, then the pair of losses is defined as the vector $\mathbf{X}_2 = (X_1, X_2)$, and let U be a uniform random variable over the interval $[0, 1]$.

Example 4.3.1. Let $X_1 = 2U$ and $X_2 = 1 + U$. The aggregate loss is given by $S_2 = 3U + 1$ and $S_2 \in [1, 4]$. The CDF of each agent is given by $F_{X_1}(x) = \frac{x}{2}$, for any $x \in [0, 2]$ and $F_{X_2}(x) = x - 1$, for any $x \in [1, 2]$, respectively. By taking the inverse of CDF, we get the quantile function for each agent $F_{X_1}^{-1}(p) = 2p$ and $F_{X_2}^{-1}(p) = 1 + p$, respectively. Hence, the comonotonic modification of $\mathbf{X}_2^c = (F_{X_1}^{-1}(U), F_{X_2}^{-1}(U)) = (2U, U + 1)$

It is straightforward to calculate the comonotonic sum $S_2^c = 3U + 1$. Its distribution function and quantile function are given by $F_{S_2^c}(s) = \frac{s-1}{3}$ for any $s \in [1, 4]$. The result would be the quantile level both agents have to abide by in the scheme.

To calculate the exact amount through pooling, we solve for $F_{X_1}^{-1}(F_{s_2^c(S_2)}) = 2(\frac{S_2-1}{3}) = 2U$ and $F_{X_2}^{-1}(F_{s_2^c(S_2)}) = (\frac{S_2-1}{3}) + 1 = U + 1$. Notice that the amount after pooling ended up the same amount for individual loss payment before pooling. This result satisfies the definition of actuarial fairness such that the payment amounts before and after pooling stay the same. Additionally, notice that the transformation g_i is linear in this example, this further proves the fairness defined in the previous section.

Example 4.3.2. Let $X_1 = 2U$ and $X_2 = 1 - U/2$. The aggregate loss is given by $S_2 = \frac{3}{2}U + 1$ and $S_2 \in [1, \frac{5}{2}]$. The CDF of each agent is given by $F_{X_1}(x) = \frac{x}{2}$, for any $x \in [0, 2]$ and $F_{X_2}(x) = 2x - 1$, for any $x \in [\frac{1}{2}, 1]$ respectively. The quantile functions are given by $F_{X_1}^{-1}(p) = 2p$ and $F_{X_2}^{-1}(p) = \frac{p+1}{2}$, respectively. Notice that the transformation is nonlinear in this example, it is reasonable to expect the scheme to not be actuarially fair.

It is straightforward to verify that the comonotonic sum is $S_2^c = \frac{5U+1}{2}$ and the quantile level in the scheme is given by $F_{S_2^c}(s) = \frac{2s-1}{5}$ for any $s \in [1/2, 3]$. The exact amount through pooling is given by $F_{X_1}^{-1}(F_{s_2^c(S_2)}) = \frac{6U+2}{5}$ and $F_{X_1}^{-1}(F_{s_2^c(S_2)}) = \frac{3U+6}{10}$.

5 Additional structures

In this section, we are interested in other aspects of the risk sharing model.

5.1 Sustainable pool of contributions/premiums

In order to be a member of the pool, participants are required to contribute a fixed, regular or periodic premiums C_i over the T period that the pool operates. Even though the premiums are paid to the common fund as a fee to protect participants from big losses in the future, there may be a long gap between contributing and actually receiving benefits at the time of losses. That leads to cases where participants withdraw from the pool to avoid paying for the losses that other participants may incur. Collectively, these actions can be detrimental to the whole pool as premiums increase for the remaining high-risk agents. To maintain a sustainable and stable pool of risk-sharing agents, the premiums can be considered as a form of investment with penalties for early contract termination. If the losses are expected, a termination payment can offset the costs, ensuring fairness in the pool. Another aspect to consider is the heterogeneity among participants. Vriens et al. [2022] found that more heterogeneous groups are better at coping with instability and short-term cost oscillations than more homogeneous ones. This suggests that C_i should be proportional to the level of risk of each participant i. In other words, $C_i \propto X_i$. In addition, before expected large losses, high-risk individuals tend to join the pool to benefit from the support. Hence, the pool can require additional upfront contributions (entry fees) that reflect the current risk level and surplus of the pool.

5.2 Addition and Removal of Agents

While the local redistribution properties of fair merging and fair splitting touch upon the removal and addition of risks, they do so in specialized contexts. Fair merging applies only to a specific type of removal where one participant's loss becomes zero because it has been entirely transferred to another participant already in the pool. Fair splitting applies to only a specific type of addition where a new participant is created specifically to take on a portion of an existing member's loss. We now consider the broader implication of expanding or reducing a risk pool through the entry or removal of a new member with their own separate risk.

5.2.1 Addition of agents

Suppose we begin with a risk-sharing pool of n participants with an individual loss vector $X = (X_1, \ldots, X_n)^T$ and an allocation rule $h(X) = (h_1(X), \ldots, h_n(X))^T$ satisfying the full-allocation property

$$\sum_{i=1}^{n} h_i(X) = \sum_{i=1}^{n} X_i.$$

A new participant joins, $n+1$, joins the pool with an individual loss X_{n+1}. The new loss vector becomes $X' = (X_1, \ldots, X_n, X_{n+1})^T$, and the aggregate loss increases to $S'_{n+1} = \sum_{j=1}^{n+1} X_j$. Thus, we have the new allocation vector $h'(X') = (h'_1(X'), \ldots, h'_{n+1}(X'))^T$, which redistributes the total loss S'_{n+1} among the $n+1$ participants. By the full-allocation property:

$$\sum_{j=1}^{n+1} h'_j(X') = \sum_{j=1}^{n+1} X_j.$$

An example is the linear rule $h(X) = AX$, where A is a column stochastic matrix with $\sum_i \alpha_{ij} = 1$ for each j. If we start with $n = 3$, the rule is:

$$h(X) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \quad \text{with} \sum_{i=1}^{3} \alpha_{ij} = 1.$$

If a fourth agent joins with loss X_4, we must expand the matrix A to a new 4×4 column stochastic matrix A' to accommodate the new participant and loss. The new coefficients α'_{ij} must be defined to ensure this property is preserved. The specific method for defining these new coefficients depends on the pool's design principles, but the resulting risk-sharing rule becomes:

$$h'(X') = \begin{bmatrix} \alpha'_{11} & \alpha'_{12} & \alpha'_{13} & \alpha'_{14} \\ \alpha'_{21} & \alpha'_{22} & \alpha'_{23} & \alpha'_{24} \\ \alpha'_{31} & \alpha'_{32} & \alpha'_{33} & \alpha'_{34} \\ \alpha'_{41} & \alpha'_{42} & \alpha'_{43} & \alpha'_{44} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix} \quad \text{with} \sum_{i=1}^{4} \alpha'_{ij} = 1.$$

The static analysis above assumes the pool's composition changes at the boundary of an observation period. In a real, ongoing P2P system operating over T periods, a new participant i joining at an intermediate time $t < T$ raises questions regarding their initial contribution, C_i. The issue is how to treat the pool's accumulated financial history. If past contributions have exceeded past losses, the pool holds a surplus. We need to consider how a new member interacts with this surplus:

- Standard Contribution: The new participant pays the standard contribution for the current period, without any adjustment for the pool's history. They immediately benefit from the stability provided by any existing surplus. This may be perceived as unfair by the original members.
- Contribution with a Buy-In Fee: To ensure fairness, the new participant is required to pay an initial contribution plus a "buy-in" fee. This fee would represent their share of the accumulated surplus, bringing their net position in line with that of existing members.
- NAV-Based Contribution: The "buy-in" is handled automatically. The new member's contribution buys them a number of "shares" at the current net asset value, which already reflects the accumulated surplus or deficit, ensuring they pay the current fair price.

Mathematically, with a participant joining at time $\tau \in (0, 1]$, the input vector for the allocation rule becomes, $X' = (X_1^{(0,1]}, \ldots, X_n^{(0,1]}, X_k^{(\tau,1]})$, a set of random variables that are not identically distributed. This violates the model's implicit assumption that the input vector consists of temporally homogeneous random variables, and hence making the allocation function h is ill-defined. Therefore, the resulting redistribution of risk and contributions depends on the specific operational rules the pool uses to handle such mid-period entries.

5.2.2 Removal of agents

In the simplest case, a participant leaves before claiming any support from the pool. Consider removing a participant i from the pool by setting their loss to 0, define: $X = (X_1, \ldots, X_{i-1}, 0, X_{i+1}, \ldots, X_n)^T$. Because $X_i = 0$, participant i incurs no loss and withdraws from the risk-sharing pool. Under the full-allocation rule, the total pooled loss is unchanged since other X_j remain the same but the share from i must now be reallocated to the remaining $n-1$ participants. Thus, we have the new allocation vector $h^{(\backslash i)} = (h_1^{(\backslash i)}, \ldots, h_{i-1}^{(\backslash i)}, h_{i+1}^{(\backslash i)}, \ldots, h_n^{(\backslash i)})$, which redistributes the total loss $\sum_{j \neq i} X_j$ among $n-1$ participants. Then, by the full-allocation rule,

$$\sum_{j \neq i} h_j^{(\backslash i)}(X) = \sum_{j \neq i} X_j.$$

An example is the linear rule $h(X) = BX$, where B is any column stochastic matrix with $\sum_i \alpha_{ij} = 1$ for each j. If $n = 3$, the linear risk-sharing rule reduces to

$$h(X) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \text{ with} \sum_{i=1}^{3} \alpha_{ij} = 1$$

If agent 3 leaves at time $t = 2$, we have to drop the α_{i3} which means the stochastic property no longer preserves. Hence, one solution is to normalize the remaining coefficients: $\alpha'_{ij} = \frac{\alpha_{ij}}{\sum_{k=1}^{2} \alpha_{kj}}$. The risk-sharing rule now becomes:

$$h^{(\setminus 3)}(X) = \begin{bmatrix} \alpha'_{11} & \alpha'_{12} \\ \alpha'_{21} & \alpha'_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ with} \sum_{k=1}^{2} \alpha'_{ij} = 1$$

In the case where we remove a person who has contributed to the pool but has not received compensation for (potential) losses. In particular, the pool operates over T periods. Participant i is active until time $t < T$, then leaves. Participant i made contribution C_i during the periods $1 : t$, but have not received compensation for events in $(t + 1) : T$ because of their withdrawal from the pool. We need to consider issues related to the accummulated contributions they made to the pool before their withdrawal. If

- C_i is premiums and if participant i didn't claim, they don't get refunded. C_i is added to the pool and used to pay for future claims of other participants.
- C_i is fully refunded. People can exploit by exiting before big losses that won't affect them.
- C_i is partially refunded: participants can withdraw and get a refund for the remainder of the period they've paid for. The rest of your premiums (which you paid before cancelling) go towards paying claims for the remaining participants. This is the policy of companies such as Lemonade, a kind of Peer-to-Peer insurer.

Now, consider the case where participant i has received some benefit B_i as some losses happen during the period $i : t$. The net balance before they exit is $NB_i = \sum_{s=1}^{t} (C_i^s - B_i^s)$ where s can be any payment period during the multi-period cycle. We have two scenarios:

- If $NB_i > 0$, participant i could negotiate partial payout or just leave the remaining C_i in the pool which is then redistributed and exit without refunds.
- If $NB_i < 0$, participant i received more than they contributed so no refund. This can be abused for expected large losses.

Under different risk-sharing rules, the redistribution of contributions vary.

6 Future work

6.1 The FEMA flood data

The National Flood Insurance Program (NFIP) was established through the National Flood Insurance Act of 1968 to protect lives and property from flood damage. It is aimed to reduce the burden on the nation's resources rather than with any expectation of profitability.

Open access data from the National Flood Insurance Program is available through `https://www.fema.gov/about/openfema/data-sets`. A summary statistics of our numerical variables are provided in Table 1. Table 3 illustrates the mean claims value by state while table 2 shows the mean of claims paid in dollar values by state. Our study opens several directions for future exploration using this FEMA flood data. A natural extension is to investigate state-specific risk factors, such as flood zones, building codes, and mitigation efforts to develop regionally customized insurance pricing models using FEMA flood maps and historical claim data. More advanced clustering methods could be applied to uncover subtler patterns in claim and coverage data.

Table 1: Summary Statistics of Main Numerical Variables

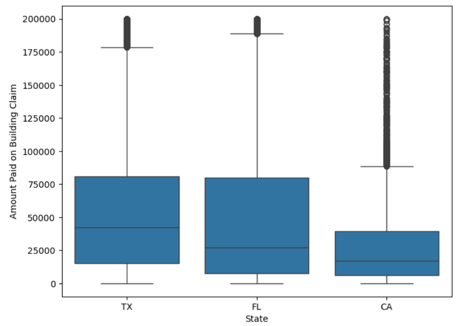
Variable	No. of obs.	Mean	SD	Min	Max
amountPaidOnBuildingClaim	775117	54861.96	75717.50	0.00	10741476.00
amountPaidOnContentsClaim	775117	16524.85	30329.09	0.00	750000.00
amountPaidOnIncreasedCostOfComplianceClaim	775117	817.42	4626.86	0.00	60000.00
totalBuildingInsuranceCoverage	775117	207880.60	1099448.00	0.00	243903000.00
totalContentsInsuranceCoverage	775117	54988.90	56242.84	0.00	3000000.00
buildingDamageAmount	775117	59300.07	667801.50	0.00	555560000.00
netBuildingPaymentAmount	775117	54810.43	75666.23	0.00	10741476.00
buildingPropertyValue	775117	1697410.00	41336310.00	0.00	2063468000.00
contentsDamageAmount	775117	21757.52	89885.96	0.00	19230510.00
netContentsPaymentAmount	775117	16493.62	30293.95	0.00	500000.00
contentsPropertyValue	775117	35814.54	520247.40	0.00	281895300.00
iccCoverage	775117	28545.95	4078.55	0.00	30000.00
netIccPaymentAmount	775117	814.91	4618.88	0.00	60000.00
buildingReplacementCost	775117	2090920.00	48601130.00	0.00	2128050000.00
contentsReplacementCost	775117	3867.84	53622.98	0.00	20000000.00

Table 2: Mean of Claim Payouts by State in dollar value

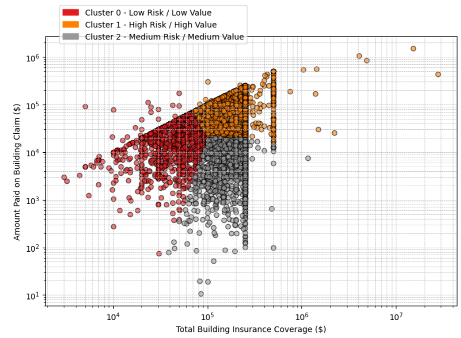
State	Total Claims	State	Total Claims	State	Total Claims	State	Total Claims
AK	51264.24	AZ	40186.11	CO	54065.32	CT	47301.62
AL	52079.96	CA	43436.50	DC	32605.44	DE	37991.87
AR	50197.95	AS	10447.53	FL	74903.58	GA	44916.23
GU	27371.08	HI	67142.84	IA	63999.28	ID	27318.93
IL	29114.75	IN	35889.60	KS	43518.81	KY	34562.70
LA	86834.26	MA	30590.48	MD	35703.17	ME	36955.45
MI	31788.40	MN	35136.97	MO	48933.68	MS	104186.47
MT	38345.03	NC	41522.58	ND	69120.93	NE	58114.53
NH	35814.68	NJ	62937.61	NM	46268.29	NV	71685.48
NY	68581.99	OH	32566.66	OK	49362.56	OR	43327.59
PA	40218.57	PR	17429.47	RI	51290.08	SC	44093.33
SD	43986.77	TN	54470.55	TX	89200.45	UN	29414.10
UT	21680.97	VA	28312.09	VI	43016.20	VT	66700.47
WA	46398.86	WI	33832.48	WV	33032.29	WY	62121.88

Table 3: Average Claim Payouts by State in dollar value

State	**Building**	**Contents**	**ICC**	**State**	**Building**	**Contents**	**ICC**
AK	44775.20	6282.17	206.87	MT	31722.73	6470.06	152.25
AL	38729.20	12588.73	762.03	NC	32302.04	8442.30	778.24
AR	39040.50	10655.35	502.10	ND	59720.74	9338.63	61.56
AS	6797.33	3650.20	0.00	NE	44833.27	12542.42	738.83
AZ	32391.67	7751.18	43.25	NH	26233.48	8428.21	1152.99
CA	34711.29	8641.54	83.67	NJ	48103.85	11959.83	2873.93
CO	43314.55	10439.80	310.97	NM	39970.86	6210.73	86.71
CT	38624.23	7740.17	937.22	NV	45732.15	25953.33	0.00
DC	28944.79	3660.64	0.00	NY	54298.85	13561.04	722.09
DE	25712.83	11348.39	930.65	OH	22911.87	9440.96	213.83
FL	60862.52	13848.27	192.79	OK	36227.70	12931.25	203.61
GA	34158.69	10448.93	308.60	OR	29669.67	11820.49	1837.43
GU	20979.79	6391.28	0.00	PA	29367.74	10322.72	528.10
HI	53741.43	13401.41	0.00	PR	9375.37	8054.10	0.00
IA	44837.17	17795.47	1366.64	RI	37770.84	13305.06	214.18
ID	21468.38	5850.55	0.00	SC	34754.07	9085.53	253.72
IL	21915.20	6140.38	1059.16	SD	34891.30	8778.84	316.63
IN	25723.48	9223.64	942.49	TN	39458.85	14741.05	270.64
KS	28073.94	15001.86	443.01	TX	65248.93	23636.43	315.08
KY	25702.40	8624.74	235.55	UN	13980.14	9433.96	6000.00
LA	64821.16	20818.83	1194.27	UT	18333.56	3347.41	0.00
MA	24606.89	5857.06	126.54	VA	21523.65	5976.31	812.12
MD	26546.60	7646.51	1510.06	VI	32478.70	10537.50	0.00
ME	31535.91	5339.08	80.47	VT	54596.41	11764.71	339.35
MI	25460.60	6156.60	171.20	WA	36285.72	9277.30	835.83
MN	27866.05	6944.50	326.42	WI	24793.81	8423.40	615.28
MO	33669.61	13826.56	1437.51	WV	21963.00	10784.04	285.25
MS	75171.25	26952.01	2063.21	WY	36717.49	25404.39	0.00



(a) Comparison of Amount Paid On Building Claim by State, for Claims < $100,000 USD



(b) KMeans Clusters of 5,000 Representative Claims

6.2 Multi-period With Agents Added

Now we consider the multi-period problem with agents added. This in return will change the size of the $A(t)$ matrix with time, as can be seen in Section 5.2. For example, if we start with three agents, then we have $A(t) \in \mathbb{R}^{3\times3}$. If a fourth agent is added at some time $t = t^*$, then $A(t^*) \in \mathbb{R}^{4\times4}$, which will be inconsistent with the calculations given in Subsection 3.2. In order to solve this problem, we propose that we fix the size of $A(t)$ to be the maximum number of possible agents, for example, if the maximum number of possible agents is five agents, then $A(t) \in \mathbb{R}^{5\times5}$, for all $t = 0, \dots, T$. However, for any $t = 0, \dots, T$, we assign zero values for the rows and columns of $A(t)$ corresponding to the inactive agents. We give a simple illustrative example as follows: Suppose we start with three agents at $t = 0$. At $t = 1$, a new agent is added. At $t = 2$, no other agents are added. Finally, at $t = 3$, a second agent was added so that we have a total number of five agents at $t = 3$. For this case, the allocation matrices will be

$$A(0) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & 0 & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & 0 & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A(1) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A(2) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & 0 \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & 0 \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & 0 \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A(3) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} & \alpha_{15} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} & \alpha_{25} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} & \alpha_{35} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} & \alpha_{45} \\ \alpha_{51} & \alpha_{52} & \alpha_{53} & \alpha_{54} & \alpha_{55} \end{bmatrix}.$$

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